APPM 2360: Midterm 3 — SOLUTIONS — FOR GRADING PURPOSES ONLY
7.00pm – 8.30pm, April 16, 2008.

Problem 1: (15 points, 3 each) State whether the following statements are (always) "TRUE" or "FALSE" (meaning not always true). You must write the full word TRUE or FALSE. For this question, you do not need to show your work or reasoning.

(a) A 3×3 real matrix A can have three complex eigenvalues.
(b) All identity matrices have the same eigenvalues independently of their dimension.
(c) If m, b, and k are real positive numbers, then any solution of the equation $m\,\ddot{x}+b\,\dot{x}+k\,x=0$ tends to zero as $t \to \infty$.
(d) An $n \times n$ real matrix A and its transpose A^{T} have the same eigenvectors.
(e) If $W(y_1, y_2)$ is the Wronskian of two functions, then $W(yy_1, yy_2) = y^2 W(y_1, y_2)$.

Solution:

(a) FALSE (complex eigenvalues of a real matrix come in complex conjugate pairs)

(b) TRUE (the only eigenvalue of an identity matrix is 1)

(c) TRUE (the solution takes the form $e^{-(b/2m)\,t}\big(c_1\,\cos(\omega\,t) + c_2\,\sin(\omega\,t)\big)$ and $-b/2m < 0$)

(d) FALSE (note that the *eigenvalues* are the same but not necessarily the eigenvectors)

(e) TRUE (the terms involving y' cancel)

Problem 2: (10 points, 2 each) Consider the equation $\ddot{y} + \dot{y} - 6\,y = f$. For each of the following choices of f, give the form of a particular solution to the equation. You do not need to motivate your answers, or determine any constants. (Sample answer: "$y_{\mathrm{p}} = \alpha\,t + \beta\,\cos(t)$ where α and β are constants".)

(a) $f(t) = e^{2\,t}$
(b) $f(t) = e^{-2\,t}$
(c) $f(t) = t\,e^t$
(d) $f(t) = \cos(2\,t) + t^2$
(e) $f(t) = t\,\sin(5\,t)$

Solution:

(a) $\alpha\,t\,e^{2\,t}$

(b) $\alpha\,e^{-2\,t}$

(c) $(\alpha + \beta\,t)\,e^t$

(d) $\alpha\,\cos(2\,t) + \beta\,\sin(2\,t) + c_0 + c_1\,t + c_2\,t^2$

(e) $(\alpha + \beta\,t)\,\sin(5\,t) + (\gamma + \delta\,t)\,\cos(5\,t)$

Problem 3: (25 points) Consider the equation $\ddot{x} + 2\,\dot{x} + 5\,x = 0$.

(a) (6 points) Construct the general solution.

(b) (6 points) Construct the solution that satisfies the initial conditions $x(0) = 1$ and $\dot{x}(0) = 3$.

(c) (7 points) Let ω be a real number and consider the equation

$$\ddot{x} + 2\,\dot{x} + 5\,x = \cos(\omega\,t).$$

Prove that a particular solution is given by $x_{\rm p} = C(\omega)\,\cos(\omega\,t - \delta)$ where $C(\omega) = 1/\sqrt{(5-\omega^2)^2 + 4\,\omega^2}$ and $\tan\delta = 2\,\omega/(5-\omega^2)$.

(d) (6 points) Find the value (or values) of ω for which the function $C(\omega)$ in part (c) attains its maximum. What happens to $C(\omega)$ as $\omega \to \infty$?

Solution:

(a) The characteristic polynomial is $r^2 + 2\,r + 5 = 0$ with roots $r_{1,2} = -1 \pm 2\,i$.

The general solution is consequently: $\boxed{x(t) = e^{-t}\big(A\,\cos(2\,t) + B\,\sin(2\,t)\big)}$

(b) The initial conditions give the following equations for A and B:

$$\begin{aligned} 1 &= x(0) = A, \\ 3 &= \dot{x}(0) = -e^{-t}\big(A\,\cos(2\,t) + B\,\sin(2\,t)\big) + e^{-t}\big(-2\,A\,\sin(2\,t) + 2\,B\,\cos(2\,t)\big) = -A + 2B \end{aligned}$$

The solution is $A = 1$ and $B = 2$ so the answer is $\boxed{x(t) = e^{-t}\big(\cos(2\,t) + 2\,\sin(2\,t)\big)}$

(c) Following the method of undetermined coefficients, we set $x_{\rm p} = A\,\cos(\omega\,t) + B\,\sin(\omega\,t)$. Then

$$\ddot{x}_{\rm p} + 2\,\dot{x}_{\rm p} + 5\,x_{\rm p} = \cdots = \big((5-\omega^2)\,A + 2\,\omega\,B\big)\,\cos(\omega\,t) + \big((5-\omega^2)\,B - 2\,\omega\,B\big)\,\sin(\omega\,t).$$

It follows that A and B must satisfy

$$\begin{bmatrix} 5-\omega^2 & 2\,\omega \\ -2\,\omega & 5-\omega^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving for A and B (using for instance the formula for the inverse of a 2×2 matrix) we find:

(1) $$A = \frac{5-\omega^2}{(5-\omega^2)^2 + 4\,\omega^2}, \qquad B = \frac{2\,\omega}{(5-\omega^2)^2 + 4\,\omega^2}.$$

Now note that

$$C\,\cos(\omega\,t - \delta) = C\,\cos(\delta)\,\cos(\omega\,t) + C\,\sin(\delta)\,\sin(\omega\,t),$$

so our solution $x_p = A\,\cos(\omega\,t) + B\,\sin(\omega\,t)$ takes the desired form if

(2) $$C\,\cos(\delta) = A, \qquad C\,\sin(\delta) = B.$$

Using the expressions for A and B in (1), we find that the solution of (2) is

$$C = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{(5-\omega^2)^2 + 4\,\omega^2}}, \qquad \tan(\delta) = \frac{B}{A} = \frac{2\,\omega}{5-\omega^2}.$$

(d) Set $f = 1/C^2$. Then C is maximized when f is minimized. We have

$$f = (5-\omega^2)^2 + 4\,\omega^2 = 25 - 10\,\omega^2 + \omega^4 + 4\,\omega^2 = 16 + 9 - 6\,\omega^2 + \omega^4 = 16 + (3-\omega^2)^2.$$

The minimum for f occurs when $\omega^2 = 3$, which is to say when $\boxed{\omega = \pm\sqrt{3}.}$

Finally we note that $\boxed{C(\omega) \to 0 \text{ as } \omega \to \infty.}$

Problem 4: (20 points) Consider the differential equation

$$\mathbf{x}' = A\,\mathbf{x}, \quad \text{where} \quad A = \begin{bmatrix} 5 & -6 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} y \\ z \end{bmatrix}.$$

(a) (7 points) Construct the eigenvalues and eigenvectors of A.

(b) (7 points) Construct the general solution $\mathbf{x}$.

(c) (6 points) The function $y = y(t)$ satisfies an equation of the form

$$y'' + a\,y' + b\,y = 0.$$

Determine the numbers a and b.

Solution:

(a) First we determine the eigenvalues. The characteristic equation is

$$0 = (5-\lambda)\,(-\lambda) + 6 = \lambda^2 - 5\,\lambda + 6.$$

The solutions are $\lambda_1 = 2$ and $\lambda_2 = 3$.

The eigenvector $\mathbf{v}_1$ associated with λ_1 satisfies $\left[\begin{array}{cc|c} 3 & -6 & 0 \\ 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$. So $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

The eigenvector $\mathbf{v}_2$ associated with λ_2 satisfies $\left[\begin{array}{cc|c} 2 & -6 & 0 \\ 1 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$. So $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

(b) The general solution is

$$\mathbf{x} = c_1\,e^{\lambda_1 t}\,\mathbf{v}_1 + c_2\,e^{\lambda_2 t}\,\mathbf{v}_2 = c_1\,e^{2t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\,e^{3t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} =$$

(c) Writing the equation $\mathbf{x}' = A\,\mathbf{x}$ in component form we find that

$$\begin{aligned} y' &= 5\,y - 6\,z \\ z' &= y. \end{aligned}$$

From these equations it follows immediately that

$$y'' = 5\,y' - 6\,z' = 5\,y' - 6\,y,$$

and so

$$y'' - 5\,y' + 6\,y = 0.$$

We see that $\boxed{a = -5 \text{ and } b = 6.}$

Problem 5: (15 points) Compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

For 5 points extra credit: Construct all eigenvalues and eigenvectors of A^{-1}.

Solution:

The characteristic equation is

$$0 = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\,(2-\lambda)\,(3-\lambda) - (2-\lambda) = (2-\lambda)\,\big((3-\lambda)^2 - 1\big)$$

$$= (2-\lambda)\,\big(8 - 6\,\lambda + \lambda^2\big) = (2-\lambda)\,(2-\lambda)\,(4-\lambda) = (2-\lambda)^2\,(4-\lambda).$$

We see that the solutions are $\lambda_1 = 4$ and $\lambda_2 = 2$.

First we construct the eigenvector(s) associated with λ_1.

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution space is one dimensional. A basis vector is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Next we construct the eigenvector(s) associated with λ_2.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The solution space is two dimensional. A basis for the solution space is

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

Extra credit problem: Recall that if A is an invertible matrix and $A\,\mathbf{v} = \lambda\,\mathbf{v}$, then $A^{-1}\,\mathbf{v} = (1/\lambda)\,\mathbf{v}$. It follows that A^{-1} has eigenvalues $1/4$ and $1/2$ and the same eigenvectors as A.

Problem 6: (15 points) Let n be a non-negative integer and consider for positive t the differential equation

$$t\,y'' - (t+n)\,y' + n\,y = 0.$$

(a) (5 points) Verify that $y_1 = e^t$ is a solution of the equation. (5p)

(b) (10 points) Find a second solution to the equation (linearly independent of the first). You may leave your answer in integral form.
Hint: You may want to look for a solution of the form $y(t) = z(t)\,e^t$.

Solution:

(a) We have

$$t\,y_1'' - (t+n)\,y_1' + n\,y_1 = t\,e^t - (t+n)\,e^t + n\,e^t = 0.$$

(b) Setting $y_2 = z\,e^t$, we find that

$t\,y_2'' - (t+n)\,y_2' + n\,y_2 = t\,(z''\,e^t + 2\,z'\,e^t + z\,e^t) - (t+n)\,(z'\,e^t + z\,e^t) + n\,z\,e^t = t\,z''\,e^t + t\,z'\,e^t - n\,z'\,e^t$

Since $t\,e^t \neq 0$ we find that z must satisfy the equation

$$z'' + \left(1 - \frac{n}{t}\right)\,z' = 0. \tag{3}$$

We solve (3) using an integrating factor. Set $f = 1 - (n/t)$. Then $F = \int f = t - n\,\log(t) = t + \log(1/t^n)$ and so $e^F = e^t/t^n$. Now (3) can be written

$$\frac{d}{dt}\left(e^F\,z'\right) = 0.$$

Integrating, we obtain $e^F\,z' = C_1$ where C_1 is a factor of integration, and so

$$z' = \frac{C_1}{e^F} = C_1\,t^n\,e^{-t}.$$

Integrating once more, we obtain

$$z = C_1\int_0^t s^n\,e^{-s}\,ds + C_2.$$

Finally, we use that $y_2 = z\,e^t$ and set $C_1 = 1$ and $C_2 = 0$ to obtain the answer

$$y_2 = \int_0^t s^n\,e^{t-s}\,dx.$$