Problem 1

a.) $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}$ 3×2 $\qquad B = \begin{bmatrix} -1 & 0 & -3 \\ 0 & 4 & 4 \end{bmatrix}$ 2×3

i.) AB: Defined

ii.) A^2: Not

iii.) $A^T B$: Not

iv.) BB^T: Defined

v.) $A + B^T$: Defined

b.) $C = \begin{bmatrix} 3 & a \\ 0 & b \end{bmatrix}$; To have only one evec we have only one eval with multiplicity, So $b = 3$.

$$\Rightarrow C = \begin{bmatrix} 3 & a \\ 0 & 3 \end{bmatrix} \text{ and } \lambda_1 = \lambda_2 = 3$$

$$\left[\begin{array}{cc|c} 3-3 & a & 0 \\ 0 & 3-3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 0 & a & 0 \\ 0 & 0 & 0 \end{array}\right]$$

: if $a = 0$ then we get 2 free variable and thus 2 evecs.
And $a \neq 0$ gives $x_2 = 0$ if $\vec{v}_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\Rightarrow \vec{v}_1 = c \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{\text{So } b = 3 \text{ and } a \neq 0}$$

c.) we need D not invertible or $|D| = 0$

So Ex $D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $E = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ $F = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

$$\Rightarrow DE = DF = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$$

d.) $|G^2| = |G|^2 = \left|\begin{bmatrix} 2 & -3 \\ 6 & -1 \end{bmatrix}\right| = -2+18=16 \Rightarrow |G| = \sqrt{16} = 4$

$|G^3| = |G|^3 = \begin{vmatrix} -2 & -5 \\ 8 & -7 \end{vmatrix} = 14+40 = 54 \Rightarrow |G| = \sqrt[3]{54} \neq 4$

So its not possible for $|G|=4$ and $|G| = \sqrt[3]{54}$

Problem 2.) $A(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

a) $|A| = \cos^2 t + \sin^2 t = 1$ 5

b) i) $W[\cos t, \sin t] = |A| = 1 \neq 0$ so $\cos t$ and $\sin t$ are linearly independent functions. 3

ii) Since $|A| \neq 0$, A^{-1} exist for all t.

So $A\vec{x} = \vec{b}$ has a unique soln for all $b \in \mathbb{R}^2$ so the cols of A span $\mathbb{R}^2$ for any value of t. 2

c) $\begin{vmatrix} \cos t - \lambda & \sin t \\ -\sin t & \cos t - \lambda \end{vmatrix} = \cos^2 t - 2\lambda \cos t + \lambda^2 + \sin^2 t$

$= 1 - 2\lambda \cos t + \lambda^2 = 0$ 3

$\lambda = \dfrac{2\cos t \pm \sqrt{4\cos^2 t - 4}}{2} = \cos t \pm \sqrt{\cos^2 t - 1}$

$= \cos t \pm \sqrt{-\sin^2 t} = \cos t \pm i \sin t$

$\boxed{\begin{aligned} \lambda_1 &= \cos t + i \sin t \\ \lambda_2 &= \cos t - i \sin t \end{aligned}}$ 2

d) $A(\tfrac{\pi}{2}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $\boxed{\begin{aligned} \lambda_1 &= i \\ \lambda_2 &= -i \end{aligned}}$ 1

$\lambda_1 = i \Rightarrow \left[\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right] \xrightarrow{iR_1 + R_2 \to R_2} \left[\begin{array}{cc|c} -i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow -ix_1 + x_2 = 0$

$x_1 = C$

$\rightarrow x_2 = Ci$

$\Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$ 4

Problem 3

a) $5 = a - b + c$ 10

$0 = a + b + c$

$3 = 9a + 3b + c$

b) $\left[\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 1 & 1 & 1 & 0 \\ 9 & 3 & 1 & 3 \end{array}\right] \xrightarrow[-9R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \begin{bmatrix} 1 & -1 & 1 & 5 \\ 0 & 2 & 0 & -5 \\ 0 & 12 & -8 & -42 \end{bmatrix}$

$\xrightarrow[-6R_2+R_3 \to R_3]{2R_1+R_2 \to R_1} \begin{bmatrix} 2 & 0 & 2 & 5 \\ 0 & 2 & 0 & -5 \\ 0 & 0 & -8 & -12 \end{bmatrix} \xrightarrow[-\frac{1}{4}R_3 \to R_3]{\frac{1}{4}R_3+R_1 \to R_1} \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & -5 \\ 0 & 0 & 2 & 3 \end{bmatrix}$

$\xrightarrow[\frac{1}{2}R_3]{\frac{1}{2}R_1,\ \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -5/2 \\ 0 & 0 & 1 & 3/2 \end{bmatrix}$ 5

2 pts RREF

c) $a = 1$

$b = -5/2 \quad \rightarrow \quad \boxed{y = x^2 - \frac{5}{2}x + \frac{3}{2}}$ 5

$c = 3/2$

Problem 4

a) if $\vec{u}, \vec{v} \in W$ and $c \in \mathbb{R}$ then

1) $\vec{u} + \vec{v} \in W$

2) $c\vec{u} \in W$

b)

i) Yes

ii) No, counter example

$$A_1 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$|A_1| = 0$ and $|A_2| = 0$

But $|A_1 + A_2| = \left|\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}\right| = 6 - 9 = -3 \neq 0$

iii) No because $f = \vec{0} \notin W_3$

Problem 5

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 3 & 4 & 5 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & 4-\lambda & 0 \\ 3 & 4 & 5-\lambda \end{vmatrix} = (5-\lambda)\left((1-\lambda)(4-\lambda)-4\right)$$ 5 pts

$$= (5-\lambda)(4-5\lambda+\lambda^2-4) = (5-\lambda)\lambda(-5+\lambda)$$

$$= -\lambda(\lambda-5)^2$$ 5 ~~10~~ pts

$$\lambda_1 = 0 \qquad \lambda_{2,3} = 5$$

$$\lambda_1 = 0 \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 3 & 4 & 5 & 0 \end{array}\right] \xrightarrow{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 0 \end{array}\right]$$ 3

$x_1 + 2x_2 = 0$ Choose $x_2 = c \to x_1 = -2c$

$3x_1 + 4x_2 + 5x_3 = 0$ $\to -6c + 4c + 5x_3 = 0$

$\to x_3 = \dfrac{2c}{5}$

$$\to \quad \vec{V}_1 = c_1 \begin{bmatrix} -10 \\ 5 \\ 2 \end{bmatrix}$$ 2

$$\lambda_2 = \lambda_3 = 5 \Rightarrow \left[\begin{array}{ccc|c} -4 & 2 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{array}\right] \xrightarrow{2R_2+R_1=R_1} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{array}\right]$$ 3

$$\xrightarrow{-3R_2+2R_3\to R_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 11 & 0 & 0 \end{array}\right] \longrightarrow$$

$x_3 = c$

$11x_2 = 0 \to x_2 = 0$

$2x_1 - x_2 = 0 \to x_1 = 0$

$$\boxed{\vec{V}_2 = c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$ 2