Sum 2008: Exam III solutions

1.) a.) Conservative ; Fits the form of : $m\ddot{\theta} + \frac{dV}{d\theta} = 0$

with $V = -\cos\theta$

b.) $T = 2\pi\sqrt{\frac{m}{k}} \rightarrow 6 = 2\pi\sqrt{\frac{9}{k}} = \frac{6\pi}{\sqrt{k}} \rightarrow k = \left(\frac{6\pi}{6}\right)^2 = \pi^2 \frac{N}{m}$

c.) $\dddot{x} + 5\dot{x} = x^3$

$$y_1 = x$$
$$y_2 = \dot{x} = \dot{y}_1$$
$$y_3 = \ddot{x} = \dot{y}_2$$
$$\dot{y}_3 = \dddot{x} = x^3 - 5\dot{x} = y_1^3 - 5y_2$$

$$\longrightarrow \boxed{\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= y_3 \\ \dot{y}_3 &= y_1^3 - 5y_2 \end{aligned}}$$

d.) $r^2 + 2ar + a^2 = 0$

$$r = \frac{-2a \pm \sqrt{4a^2 - 4a^2}}{2} = -a$$

$$x = C_1 e^{-at} + C_2 t e^{-at}$$

e.) $y_1 = \frac{1}{2}t^2$ $\qquad$ $y_2 = t^2 + 2$

$y_1' = t$ $\qquad$ $y_2' = 2t$

$y_1'' = 1$ $\qquad$ $y_2'' = 2$

$\rightarrow 1 + q\left(\frac{1}{2}t^2\right) = f$ $\qquad$ $2 + q(t^2 + 2) = f$

$$\begin{array}{r} 2 + q(t^2+2) = f \\ -1 + -\frac{1}{2}t^2 q = -f \\ \hline 1 + \left(2 + \frac{1}{2}t^2\right)q = 0 \end{array}$$

$$q = \frac{-1}{2 + \frac{1}{2}t^2}$$

$$\rightarrow f = 1 + \left(\frac{-1}{2 + \frac{1}{2}t^2}\right)\frac{1}{2}t^2$$

$$= 1 - \frac{t^2}{4 + t^2} = \boxed{\frac{4}{4+t^2} = f(t)}$$

2.) $y'' + 2y' - 3y = f(t)$

a.) $r^2 + 2r - 3 = 0$

$(r+3)(r-1) = 0$ 5 pts

$r_1 = -3 \quad r_2 = 1 \rightarrow y_h(t) = c_1 e^{-3t} + c_2 e^{t}$

2 2 1

b.) i.) $y_p = A_2 t^2 + A_1 t + A_0$ 10 pts

ii.) NOT

iii.) $y_p = A_1 \cos t + A_2 \sin t + B_1 \cos 5t + B_2 \sin 5t$ 1 pt

iv.) $t(A_3 t^3 + A_2 t^2 + A_1 t + A_0)e^{-3t}$ 1 pt

v.) $y_p = e^{-3t}(A\cos t + B\sin t)$

c.) $v_1' = -\dfrac{y_2 f}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \dfrac{-e^{t}(4e^{-3t}\sin t)}{\begin{vmatrix} e^{-3t} & e^{t} \\ -3e^{-3t} & e^{t} \end{vmatrix}} = \dfrac{-4e^{-2t}\sin t}{e^{-2t} + 3e^{-2t}} = -\sin t$ 8

$\rightarrow v_1 = \cos t$ 3

$v_2' = \dfrac{y_1 f}{4e^{-2t}} = \dfrac{e^{-3t}\cancel{4}e^{-3t}\sin t}{\cancel{4}e^{-2t}} = e^{-4t}\sin t$

$v_2 = \int \underset{\downarrow\, -4e^{-4t}}{e^{-4t}} \overset{-\cos t\, \uparrow}{\sin t}\, dt = -e^{-4t}\cos t - \int \underset{\downarrow\, -4e^{-4t}}{4e^{-4t}} \overset{\sin t\, \uparrow}{\cos t}\, dt$

$= -e^{-4t}\cos t - 4\left[e^{-4t}\sin t + \int 4e^{-4t}\sin t\, dt\right]$

$= -e^{-4t}\cos t - 4e^{-4t}\sin t - 16\int e^{-4t}\sin t\, dt$

$\rightarrow \int e^{-4t}\sin t\, dt = \frac{-1}{17}e^{-4t}(\cos t + 4\sin t)$ 3

$\rightarrow y_p = \cos t\, e^{-3t} - \frac{1}{17}e^{-4t}(\cos t + 4\sin t)e^{t}$ 2

$= \left(-\frac{4}{17}\sin t + \frac{16}{17}\cos t\right)e^{-3t}$

d.) $y_G = c_1 e^{-3t} + c_2 e^{t} + e^{-3t}\left(-\frac{4}{17}\sin t + \frac{16}{17}\cos t\right)$ 2

3.) $\ddot{x} + 3\dot{x} + 2x = 0$

a.) $r^2 + 3r + 2 = 0$

$(r+2)(r+1) = 0 \qquad r_1 = -2,\ r_2 = -1$

$x(t) = c_1 e^{-2t} + c_2 e^{-t}$

$\dot{x}(t) = -2c_1 e^{-2t} - c_2 e^{-t}$

$\longrightarrow \quad x(0) = 0 = c_1 + c_2$

$\dot{x}(0) = 6 = -2c_1 - c_2$

$6 = -c_1 \rightarrow \boxed{\begin{array}{l} c_1 = -6 \\ c_2 = 6 \end{array}}$

$\rightarrow x(t) = 6(-e^{-2t} + e^{-t})$

b.) $\dot{x}(t) = 12e^{-2t} - 6e^{-t} = 0 \rightarrow 2e^{-2t} = e^{-t}$

$\rightarrow \ln 2 + \ln e^{-2t} = \ln e^{-t}$

$\ln 2 - 2t = -t$

$\Longrightarrow \boxed{T = \ln(2) \text{ seconds}}$

c.) $E(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \frac{1}{2}\dot{x}^2 + x^2$

$E(0) = E(x(0), \dot{x}(0)) = E(0, 6) = \frac{1}{2}(36) = 18 \text{ J}$

$E(T = \ln 2) = E(x(\ln 2), \dot{x}(\ln 2))$

$x(\ln 2) = 6(-e^{\ln(\frac{1}{4})} + e^{\ln(\frac{1}{2})}) = 6(-\frac{1}{4} + \frac{1}{2})$

$= 3/2$

$\dot{x}(\ln 2) = 12e^{\ln(\frac{1}{4})} - 6e^{\ln\frac{1}{2}}$

$= 3 - 3 = 0$

$\rightarrow E(3/2, 0) = 0 + (3/2)^2 = (9/4)$

So Energy Loss: $18 - 9/4 = \frac{72-9}{4} = \boxed{\frac{63}{4} \text{ J}}$

4.) a.) $2\ddot{x} + 50x = -20\sin(5t) \iff \ddot{x} + 25x = -10\sin 5t$ 3 pts

b.) $2\ddot{x} + 50x = 0$

$r^2 + 25 = 0$

$r_{1,2} = \pm 5i \rightarrow x_h(t) = C_1\cos 5t + C_2 \sin 5t$

3 2

c.) $x_p = At\cos 5t + Bt\sin 5t = t(A\cos 5t + B\sin 5t)$ 2 $\frac{2}{10}$ w/out t

10pts

$x_p' = A\cos 5t + B\sin 5t + t(-5A\sin 5t + 5B\cos 5t)$

$= (A + 5Bt)\cos 5t + (B - 5At)\sin 5t$

$x_p'' = 5B\cos 5t - 5(A + 5Bt)\sin 5t - 5A\sin 5t + 5(B - 5At)\cos 5t$

$= (10B - 25At)\cos 5t + (-10A - 25Bt)\sin 5t$ 2

$x_p'' + 25x_p = (10B - 25At)\cos 5t - (10A + 25Bt)\sin 5t$

$+ 25At\cos 5t + 25Bt\sin 5t = -10\sin 5t$

$\rightarrow 10B\cos 5t - 10A\sin 5t = -10\sin 5t$

$\rightarrow A = 1, B = 0$

2 2

$\rightarrow x_p = t\cos(5t)$

2

d.) $x(0) = -1$

$\dot{x}(0) = 1$

$x_G = C_1\cos 5t + C_2\sin 5t + t\cos 5t$

$\dot{x} = -5C_1\sin 5t + 5C_2\cos 5t + \cos 5t - 5t\sin 5t$

$x(0) = C_1 = -1$ 2

$\dot{x}(0) = 5C_2 + 1 = 1 \rightarrow C_2 = 0$ 2

$\rightarrow x(t) = -\cos 5t + t\cos 5t$ 1

$= (t-1)\cos(5t)$

e.) Resonance, oscillations with increasing amplitude

1 1