APPM 2360 Final Exam Summer 2008

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) your lecture section, (4) your instructor's name and (5) a grading table. You have 90 minutes to work all 5 problems on the exam. Each problem is worth 20 points. Show ALL of your work in the bluebook and $\boxed{\text{box}}$ in final answers. Start each problem on a new page. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes and calculators are NOT permitted. One letter size (8.5″ × 11″) crib sheet with anything hand written on both sides is allowed.

1. For each question indicate whether you think the answer is TRUE or FALSE (i.e. not always correct). No explanations are needed.

   a) If A is a 4×3 matrix, then the linear system $A\mathbf{x} = [0\ 0\ 0\ 0]^{\mathsf{T}}$ always has a solution.

   b) Let A be an $n \times n$ matrix with n linearly independent columns. Then the reduced row echelon form of A is the $n \times n$ identity matrix.

   c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{bmatrix}$

   d) Let $A = \begin{bmatrix} a & -2 \\ 3 & 4a \end{bmatrix}$. Then there are no values of $a \in \mathbb{R}$ such that $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a saddle type equilibrium solution of $\mathbf{x}' = A\mathbf{x}$.

   e) The system, $\ddot{z} - 7z = \sin(t)$, is conservative.

   f) The initial value problem, $y' = 4y^{\frac{2}{3}}$, $y(0) = 0$, has a unique solution.

   g) The differential equation, $y' = \mathrm{e}^{y - t^2 - 1}$, is separable.

   h) $y'' + \mathrm{e}^t y = \sqrt{t}$ is a linear differential equation.

   i) $y(t) = \mathrm{e}^{t^2}$ is a solution to $y'' - 2ty' - 2y = 0$.

   j) The general solution to $\ddot{x} + 2\dot{x} + 10 = 5\cos(3t)$ has oscillations that become unbounded as $t \to \infty$.

2. Consider the matrix, $A = \begin{bmatrix} k & 4 \\ -1 & 1 \end{bmatrix}$.

   a) Find all values of k such that A has a repeated eigenvalue.

   b) Find all values of k such that the linear system, $\mathbf{x}' = A\mathbf{x}$, corresponds to a center.

   c) Find all values of k such that the columns of A form a linearly dependent set.

   d) For each value of k determined in part (c), find all vectors, $\mathbf{v}$, such that $AA^{\mathsf{T}}\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

3. Consider the following second order, differential equation: $y'' - y' = \mathrm{e}^{-t}$.

   a) Find the homogeneous solution.

   b) Find the particular solution using the method of undetermined coefficients.

   c) Find the particular solution using variation of parameters.

4. a) Find all $x(t)$ and $y(t)$ such that:
$$\begin{aligned} x' &= x + y\,, \\ y' &= -x + 3y\,. \end{aligned}$$

   b) Find all $z(t)$ such that, $z' + 2tz = t$.

— **over** —

5. Match each of the following five systems of differential equations, (1)-(5), with its phase portrait, A-G. No explanations are needed. Note: there are two extra phase portraits.

(1)
$$\begin{aligned} x' &= 4x - 5y \\ y' &= 5x - 4y \end{aligned}$$

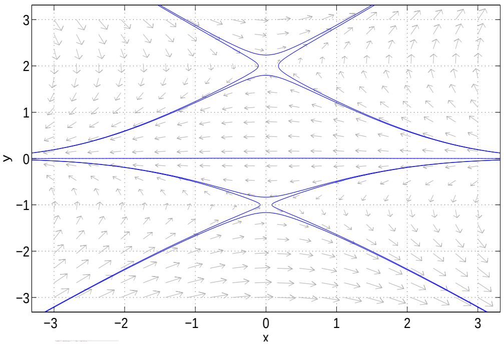
(2)
$$\begin{aligned} x' &= y \\ y' &= -y + x - x^3 \end{aligned}$$

(3)
$$\begin{aligned} x' &= (y-2)(y+1) \\ y' &= xy \end{aligned}$$

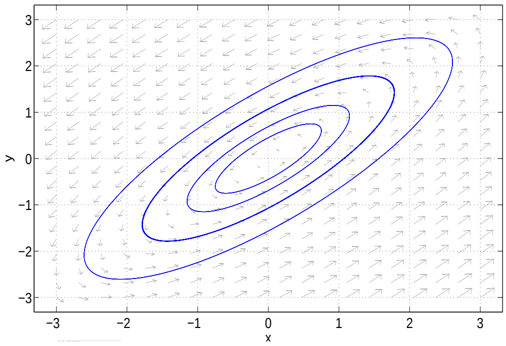
(4)
$$\begin{aligned} x' &= x + y \\ y' &= 4x + y \end{aligned}$$

(5)
$$\begin{aligned} x' &= 2x + y \\ y' &= 6x + 3y \end{aligned}$$

A



B



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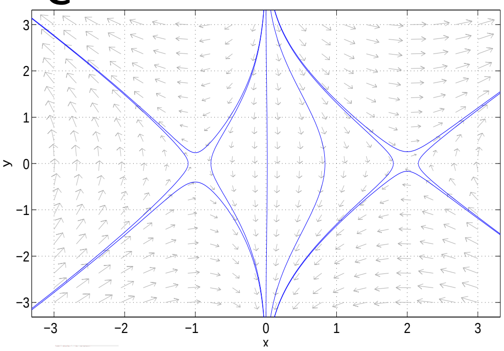

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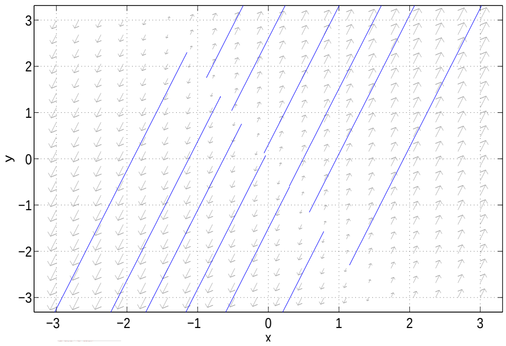

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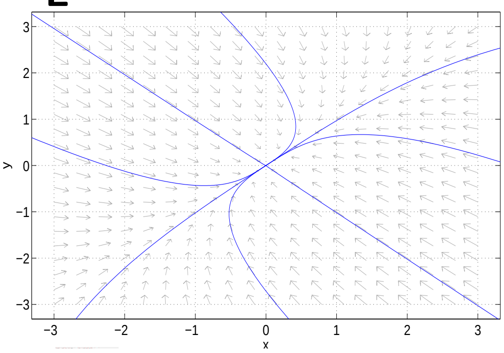

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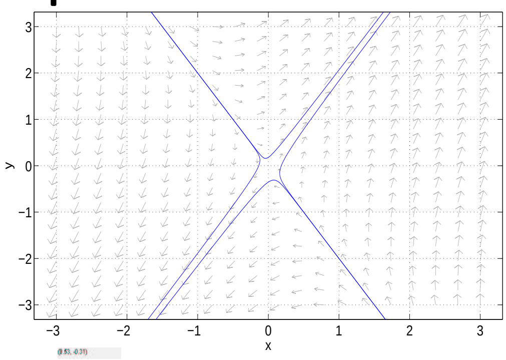

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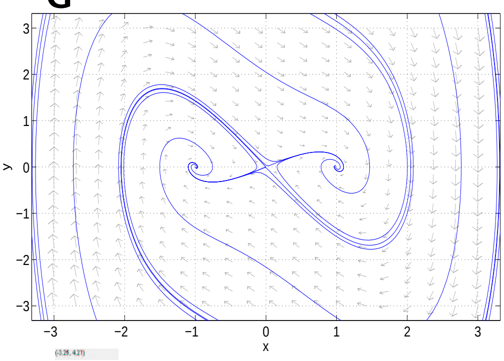
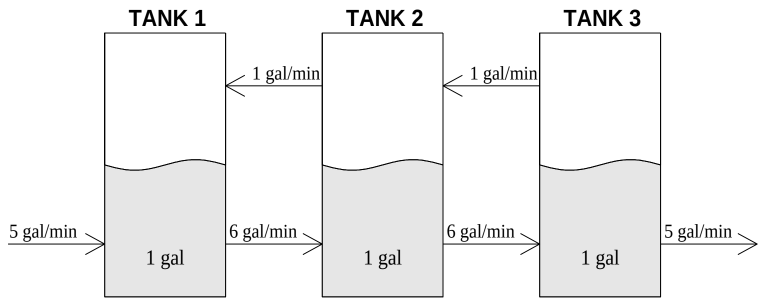


Take home problems.
Open book, closed collegue.

6. This problem concerns the three connected water tanks pictured below.



Initially tank 1 contains 1 gal of a 2 lbs/gal salt water solution, tank 2 contains 1 gal of a 3 lbs/gal salt water solution, and tank 3 contains 1 gal of fresh water. All three tanks are continuously stirred. Fresh water is poured into tank 1 at a rate of 5 gal/min, as indicted in the figure. Water is transferred between the tanks at the rates indicted in the figure. Water leaves the system via tank 3 at a rate of 5 gal/min, also as indicated in the figure.

a) Write a linear, three-dimensional, first order, differential equation system modelling the amount of salt in each tank. Use x_1, x_2 and x_3, to represent the amount of salt (in lbs) in tanks 1,2 and 3, respectively.

b) Rewrite the system you have obtained in part (a) into the vectorized form: $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Find all eigenvalues and eigenvectors of the matrix, A.

c) Determine the general solution to the system obtained in part (b).

d) Use the initial salt concentrations given above to find the exact solution to the system.

7. Scientists have recently discovered a mysterious particle called a zuon. When placed above a powerful magnet a zuon is well modelled by:

$$\begin{aligned} x' &= y - e^{2x} \\ y' &= xy - 3x \end{aligned}$$

where x is a measure of rotation of the zuon and y denotes the distance between the zuon and the magnet.

a) Determine all equilibrium solutions of this system.

b) Determine the eigenvalues of the Jacobian matrix of each equilibrium solution and use these to classify the equilibrium solutions.

c) Make a graph of the nullclines of the system. Include arrows on <u>and</u> between the nullclines showing the direction of solutions.

d) Suppose initially the rotation of a zuon is $x_0 = -1$. Using your answer from part (c), argue the fate of the distance of the zuon from the magnet as $t \to \infty$, for the following three initial values of y:

   i) $y_0 = 2$,
   ii) $y_0 = 3$,
   iii) $y_0 = 4$.