Problem 1

a.) True

b.) True

c.) False

d.) To be a saddle $|A|<0$, $|A| = 4a^2+6>0$

TRUE

e.) False

f.) False

g.) TRUE

h.) TRUE

i.) $y(t)=e^{t^2}$, $y' = 2te^{t^2}$, $y'' = 2e^{t^2} + 4t^2e^{t^2}$

$\cancel{2e^{t^2}} + \cancel{4t^2e^{t^2}} - \cancel{4t^2e^{t^2}} - \cancel{2e^{t^2}} = 0$ TRUE

j.) False $b=2>0$

Problem 2

a) $A = \begin{bmatrix} k & 4 \\ -1 & 1 \end{bmatrix}$

evals: $\begin{vmatrix} k-\lambda & 4 \\ -1 & 1-\lambda \end{vmatrix} = k - (k+1)\lambda + \lambda^2 + 4 = 0$

$$\lambda = \frac{k+1 \pm \sqrt{k^2+2k+1-16-4k}}{2}$$

repeated eval if $k^2 - 2k - 15 = 0$

or $(k-5)(k+3) = 0$

$\boxed{k=5 \text{ or } k=-3}$

b) For center we need complex evals with

$\alpha = 0 \Rightarrow \frac{k+1}{2} = 0 \Rightarrow \boxed{k=-1}$

And $\Delta = (-1)^2 - 2(-1) - 15 = 1+3-15 < 0$ ✓

c) cols of A are linearly dep if $|A| = 0$

$|A| = \begin{vmatrix} k & 4 \\ -1 & 1 \end{vmatrix} = k+4 = 0 \Rightarrow \boxed{k=-4}$

d) $k=-4$ $\quad A = \begin{bmatrix} -4 & 4 \\ -1 & 1 \end{bmatrix} \quad A^T = \begin{bmatrix} -4 & -1 \\ 4 & 1 \end{bmatrix}$

$AA^T = \begin{bmatrix} -4 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -4 & -1 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 32 & 8 \\ 8 & 2 \end{bmatrix}$

$\left[\begin{array}{cc|c} 32 & 8 & 0 \\ 8 & 2 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}R_2]{\frac{1}{8}R_1} \left[\begin{array}{cc|c} 4 & 1 & 0 \\ \cancel{4} & \cancel{1} & \cancel{0} \end{array}\right]$ $\quad 4v_1 + v_2 = 0$

Let $v_1 = c \rightarrow v_2 = -4c$

$\boxed{\vec{v} = c\begin{bmatrix} 1 \\ -4 \end{bmatrix},\ c \in \mathbb{R}}$

Problem 3 $y'' - y' = e^{-t}$

a.) $r^2 - r = 0$

$r(r-1) = 0 \quad r=0 \quad r=1$

$$\boxed{y(t) = c_1 + c_2 e^{t}}$$

b.) $y_p(t) = Ae^{-t}$

$y_p' = -Ae^{-t}$

$y_p'' = Ae^{-t}$

$\Rightarrow y'' - y' = Ae^{-t} - (-Ae^{-t}) = e^{-t}$

$\rightarrow 2Ae^{-t} = e^{-t}$

$2A = 1 \Rightarrow A = \frac{1}{2}$

$$\Rightarrow \boxed{y_p(t) = \frac{1}{2}e^{-t}}$$

c.) $v_1' = \dfrac{-y_2 f}{W[y_1, y_2]} = \dfrac{-e^{t} e^{-t}}{\begin{vmatrix} 1 & e^{t} \\ 0 & e^{t} \end{vmatrix}} = \dfrac{-1}{e^{t}} = -e^{-t}$

$\Rightarrow v_1(t) = \int e^{-t}\,dt = e^{-t}$

$v_2' = \dfrac{y_1 f}{W} = \dfrac{(1)\, e^{-t}}{e^{t}} = e^{-2t} \Rightarrow v_2(t) = \int e^{-2t}\,dt$

$= \frac{-1}{2}e^{-2t}$

$\rightarrow y_p(t) = v_1 y_1 + v_2 y_2 = e^{-t} + \frac{-1}{2}e^{-2t} e^{t}$

$= e^{-t} - \frac{1}{2}e^{-t} = \frac{1}{2}e^{-t}$

$$\boxed{y_p(t) = \frac{1}{2}e^{-t}}$$

Problem 4

a) $\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

evals $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = 3 - 4\lambda + \lambda^2 + 1$

$= 4 - 4\lambda + \lambda^2 = 0$

$(\lambda - 2)^2 = 0$

4 $\lambda = 2$ repeated

evecs: $\left[\begin{array}{cc|c} -1 & 1 & 0 \\ \cancel{-1} & \cancel{1} & \cancel{0} \end{array}\right]$ $-x_1 + x_2 = 0$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ 2

need generalized evec:

$\left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \rightarrow -x_1 + x_2 = 1$

$x_1 = c \rightarrow x_2 = 1 + c$

$\vec{u} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2

-OR- if $x_2 = c$

$\rightarrow x_1 = c - 1$

$\rightarrow \vec{u} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\left(t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ 2

$$\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} t \\ t+1 \end{bmatrix}}$$

Prob 4) b) $z' + 2tz = t$

$$\mu = e^{\int 2t\,dt} = e^{t^2} \quad 2$$

$$e^{t^2} z' + 2te^{t^2} z = te^{t^2}$$

$$\int \frac{d}{dt}\left[e^{t^2} z\right] = \int te^{t^2}\,dt \quad 4$$

$$e^{t^2} z = \frac{1}{2} e^{t^2} + C \quad 2$$

$$\boxed{z = \frac{1}{2} + Ce^{-t^2}} \quad 2$$

Prob 5

B G A F D

(1) $\begin{array}{l} x' = 4x - 5y \\ y' = 5x - 4y \end{array} = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{vmatrix} 4-\lambda & -5 \\ 5 & -4-\lambda \end{vmatrix} = -16 + \lambda^2 + 25 = 0$$

$$\lambda^2 + 9 = 0$$

$$\lambda = \pm 3i$$

So center equil: $\boxed{B}$

(2) $\begin{array}{l} x' = y \\ y' = -y + x - x^3 \end{array}$ $\begin{array}{l} \to \quad y = 0 \\ \quad\; y = x - x^3 \to x = 0 \text{ or } x = \pm 1 \end{array}$

equil: $(0,0), (1,0), (-1,0)$

$\to \boxed{G}$

(3) $\begin{array}{l} x' = (y-2)(y+1) \to y = 2 \text{ or } y = -1 \\ y' = xy \quad \to x = 0 \text{ or } y = 0 \end{array} \Big\}$ equil: $(0,2)$, $(0,-1)$

$\boxed{A}$

(4) $\begin{array}{l} x' = x + y \\ y' = 4x + y \end{array} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = 1 - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0$$

Saddles! $\boxed{F}$

(5) $\begin{array}{l} x' = 2x + y \\ y' = 6x + 3y \end{array} \to \begin{bmatrix} 2 & 1 \\ 6 & 3 \end{bmatrix} \to \begin{vmatrix} 2-\lambda & 1 \\ 6 & 3-\lambda \end{vmatrix} = 6 - 5\lambda + \lambda^2 - 6 \to \boxed{D}$

$= \lambda^2 - 5\lambda = 0$

$$\frac{y'}{y} = t$$

Problem 6

$x_1(0) = 2$
$x_2(0) = 3$
$x_3(0) = 0$

a.) $x_1' = -6x_1 + x_2$ 2 9
$x_2' = 6x_1 - 7x_2 + x_3$ 2
$x_3' = 6x_2 - 6x_3$ 2

b.) $\vec{x}' = \underbrace{\begin{bmatrix} -6 & 1 & 0 \\ 6 & -7 & 1 \\ 0 & 6 & -6 \end{bmatrix}}_{A} \vec{x}$ 3

evals: $|A - \lambda I| = \begin{vmatrix} -6-\lambda & 1 & 0 \\ 6 & -7-\lambda & 1 \\ 0 & 6 & -6-\lambda \end{vmatrix}$

$= (-6-\lambda)((-7-\lambda)(-6-\lambda) - 6) - (6(-6-\lambda))$

$= (-6-\lambda)\left[42 + 13\lambda + \lambda^2 - 6 - 6\right]$

$= -(-6-\lambda)\left[\lambda^2 + 13\lambda + 30\right] = -(6+\lambda)(\lambda+10)(\lambda+3) = 0$

$\boxed{\lambda_1 = -6,\ \lambda_2 = -10,\ \lambda_3 = -3}$ 3

evecs: $\lambda_1 = -6$ $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 6 & -1 & 1 & 0 \\ 0 & 6 & 0 & 0 \end{array}\right] \longrightarrow$ $x_2 = 0$, $6x_1 + x_3 = 0 \Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix}}$

Let $x_1 = c$
$x_3 = -6c$

2pts each evec } 6

$\lambda_2 = -10$ $\left[\begin{array}{ccc|c} 4 & 1 & 0 & 0 \\ 6 & 3 & 1 & 0 \\ 0 & 6 & 4 & 0 \end{array}\right] \xrightarrow{-2R_2 + 3R_1 \to R_2} \begin{bmatrix} 4 & 1 & 0 & 0 \\ 0 & -3 & -2 & 0 \\ 0 & 6 & 4 & 0 \end{bmatrix}$

$\rightarrow$ $4x_1 + x_2 = 0$
$3x_2 + 2x_3 = 0$

Let $x_2 = 4c \rightarrow x_1 = -c$
$x_3 = -6c \rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix}$

$\lambda_3 = -3$

$$\left[\begin{array}{ccc|c} -3 & 1 & 0 & 0 \\ 6 & -4 & 1 & 0 \\ 0 & 6 & -3 & 0 \end{array}\right] \xrightarrow{2R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} -3 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ \cancel{0} & \cancel{6} & \cancel{-3} & \cancel{0} \end{array}\right]$$

$$-3x_1 + x_2 = 0$$
$$-2x_2 + x_3 = 0$$

Let $x_2 = 3c \to x_1 = c$ $\to$ $\boxed{\vec{V}_3 = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}}$

$\to x_3 = 6c$

c.) $$\vec{X}_G(t) = C_1 e^{-6t}\begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix} + C_2 e^{-10t}\begin{bmatrix} 1 \\ -4 \\ 6 \end{bmatrix} + C_3 e^{-3t}\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

2

5 d.) $$\vec{x}(0) = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & 3 \\ -6 & 6 & 6 \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix}$$ 3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -4 & 3 & 3 \\ -6 & 6 & 6 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -4 & 3 & 3 \\ 0 & 12 & 12 & 12 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc|c} 4 & 0 & 7 & 11 \\ 0 & -4 & 3 & 3 \\ 0 & 0 & 21 & 21 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 4 & 0 & 0 & 4 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\longrightarrow \begin{array}{l} C_1 = 1 \\ C_2 = 0 \\ C_3 = 1 \end{array}$$ 2

$$\Rightarrow \vec{x}(t) = e^{-6t}\begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix} + e^{-3t}\begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$$

Problem 7

$x' = y - e^{2x} = f \qquad f_x = -2e^{2x} \quad f_y = 1$

$y' = xy - 3x = g \qquad g_x = y - 3 \quad g_y = x$

a) $x' = 0 = y - e^{2x}$ if $y = e^{2x}$

$y' = 0 = xy - 3x$ if $y = 3$ or $x = 0$

$$\boxed{(0, 1) \text{ and } (\ln\sqrt{3}, 3)}$$

b) $J(0,1) = \begin{bmatrix} -2 & 1 \\ -2 & 0 \end{bmatrix}$

evals: $\begin{vmatrix} -2-\lambda & 1 \\ -2 & -\lambda \end{vmatrix} = 2\lambda + \lambda^2 + 2 = 0$

$\lambda = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

So the evals are complex and $\alpha = -1 < 0$ so $(0,1)$ is a stable equil, with inward spiral solns close to it on the phase portrait

$J(\ln\sqrt{3}, 3) = \begin{bmatrix} -6 & 1 \\ 0 & \ln\sqrt{3} \end{bmatrix}$

evals: $\begin{vmatrix} -6-\lambda & 1 \\ 0 & \ln\sqrt{3} - \lambda \end{vmatrix} = -6\ln\sqrt{3} + (6 - \ln\sqrt{3})\lambda + \lambda^2 = 0$

$\lambda = \dfrac{\ln\sqrt{3} - 6 \pm \sqrt{36 - 12\ln\sqrt{3} + (\ln\sqrt{3})^2 + 24\ln\sqrt{3}}}{2}$

$= \frac{1}{2}\ln\sqrt{3} - 3 \pm \dfrac{\sqrt{12\ln(\sqrt{3}) + 36 + (\ln\sqrt{3})^2}}{2}$

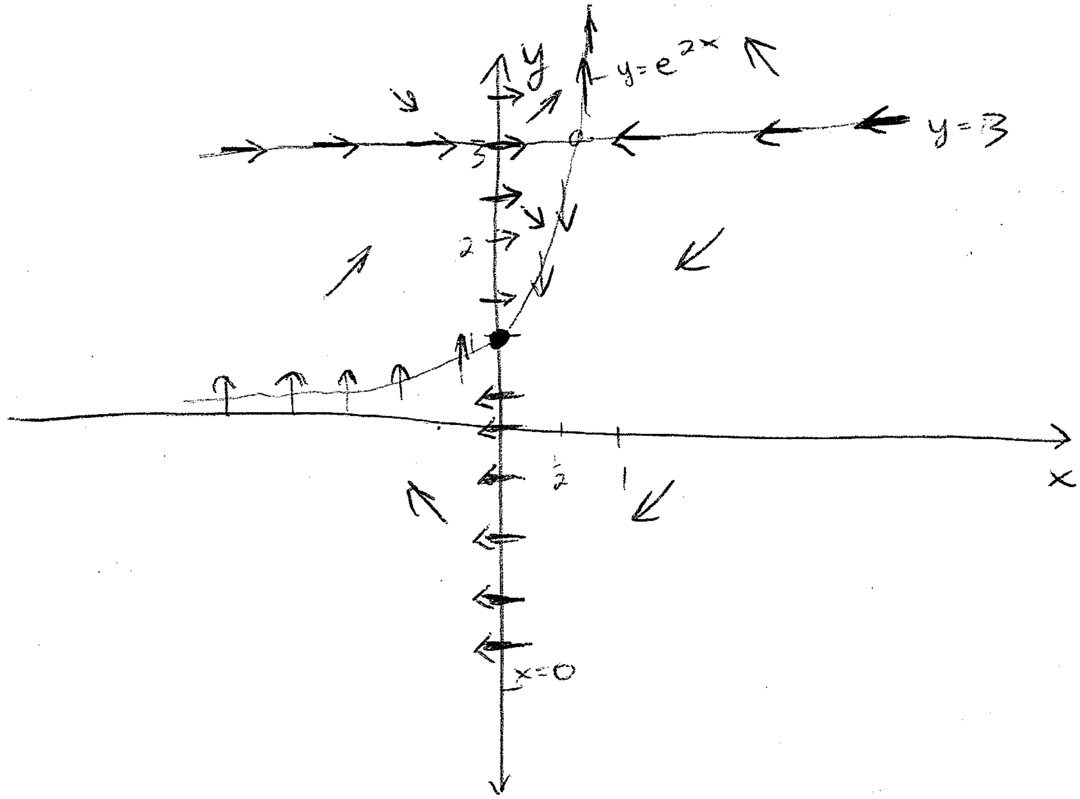
$= \frac{1}{2}\ln\sqrt{3} - 3 \pm \dfrac{\sqrt{(6 + \ln\sqrt{3})^2}}{2}$

$\lambda_1 = \ln\sqrt{3} > 0$

$\lambda_2 = -6$

So the evals are real and have opposite signs so then $(\ln\sqrt{3}, 3)$ is an unstable equil with saddle node behavior.

c)



d) i) $y_0=2,\ x_0=-1$ Soln will spiral inwards to $(0,1)$ as $t\to\infty$

ii) $x_0=-1,\ y_0=3$ Soln will follow the nullcline on $y=3$ to $(\ln\sqrt{3}, 3)$ as $t\to\infty$.

iii) $x_0=-1,\ y_0=4$ Both x and y will approach infinity as $t\to\infty$ going towards $y=e^{2x}$.