#1 a) Linear, nonhomogeneous → integrating factor

$$\mu(t) = e^{-t}$$

$$e^{-t}\left(\frac{d}{dt}y - y\right) = e^{-t}e^{3t}$$

$$\Downarrow$$

$$\frac{d}{dt}\left(e^{-t}y\right) = e^{2t}$$

$$e^{-t}y = \int e^{2t}\,dt = \tfrac{1}{2}e^{2t} + c$$

$$\Rightarrow \quad y(t) = \tfrac{1}{2}e^{3t} + ce^{t}$$

I.V.P. $y(0) = 1 = \tfrac{1}{2} + c \Rightarrow c = \tfrac{1}{2}$

$$\boxed{y(t) = \tfrac{1}{2}\left(e^{3t} + e^{t}\right)}$$

b) Put in standard form first

$$\frac{dy}{dt} + \left(3 + \frac{3}{t}\right)y = t^{-3}$$

integrating factor $\mu = e^{\int p(t)\,dt} = e^{3t + 3\ln t} = t^3 e^{3t}$

$$t^3 e^{3t}\left(\frac{d}{dt}y + \left(3 + \frac{3}{t}\right)y\right) = e^{3t}$$

$$\frac{d}{dt}\left(t^3 e^{3t} y\right) = e^{3t}$$

$$t^3 e^{3t} y = \int e^{3t}\,dt \quad \tfrac{1}{3}e^{3t} + c$$

$$y = \frac{1}{3t^3} + \frac{c}{t^3}e^{-3t}$$

$y(1) = 0 = \tfrac{1}{3} + ce^{-3} \quad \Rightarrow \quad c = -\frac{e^3}{3}$

so $y(t) = \frac{1}{3t^3}\left(1 - e^{3}e^{-3t}\right) = \boxed{\frac{1}{3t^3}\left(1 - e^{-3(t-1)}\right)}$

#2 a) Since L is linear $L(cy) = cL(y)$

so if $L(y) = 2e^{2t}$, then $L(2y) = 2L(y) = \boxed{4e^{2t}}$

Since L is linear $L(0y) = L(0) = 0L(y) = 0$

so $\boxed{L(0) = 0}$

b) Since L is linear $L(y+2z) = L(y) + L(2z)$

$= L(y) + 2L(z)$

$= 2e^{2t} + 0 = \boxed{2e^{2t}}$

c) Yes. Because $f(t,y) = \frac{\cos y}{1-y^2}$ is continuous except at $y = \pm 1$

Similarly $\frac{\partial f}{\partial y} = \frac{(\text{bigmess})}{(1-y^2)^2}$ is also continuous except at $y = \pm 1$

Since $y(0) = 0$ we are away from problems

In this case Picard's theorem implies there is a unique solution $y(t)$ with $y(0) = 0$ for some interval

$\boxed{-h < t < h}$ [we do not know h!]

d) $y_p = t^{3/2} + 1$ $\qquad \frac{dy_p}{dt} = \frac{3}{2}t^{1/2}$

$\frac{3}{2}(y-1)^{1/3} = \frac{3}{2}(t^{3/2})^{1/3} = \frac{3}{2}t^{1/2}$ $\boxed{\text{Same} \checkmark}$

note also $y(0) = 1$. ✓

Picard's theorem does not apply since $\frac{\partial f}{\partial y} = \frac{1}{2}(y-1)^{-2/3} \to \infty$ at $y=1$

so we can't guarantee a unique solution

Indeed another solution is $\boxed{y(t) = 1}$ {the equilibrium}

3 a) $\frac{dA}{dt} = 0.1A$ with $A(0) = \$10^6$

$\Rightarrow A(t) = 10^6 \cdot e^{0.1t}$

So $A(10) = 10^6 e^1 = \boxed{\$2.71 \text{ million}}$

b) $\frac{dA}{dt} = 0.1A - 10^5$

Solution is $A(t) = \frac{10^5}{0.1} + Ce^{0.1t}$

$A(0) = 10^6 \Rightarrow C = 0$ so $\boxed{A(t) = \$10^6 \text{ for all time}}$

4) a) $t^3 \frac{dy}{dt} = 2y^2$

Seperation of variables

$$\int \frac{dy}{y^2} = \int \frac{2}{t^3}\,dt$$

$$-\frac{1}{y} = \frac{-1}{t^2} + c$$

$$y(t) = \frac{1}{-c + 1/t^2} = \frac{t^2}{1-ct^2}$$

this solution does not include the equilibrium

So general solution is $\boxed{y(t) = \begin{cases} \frac{t^2}{1-ct^2} & y_0 \neq 0 \\ 0 & y_0 = 0 \end{cases}}$

b) if $y(1) = 1$ $\quad 1 = \frac{1}{1-c} \Rightarrow c = 0$

So $\boxed{y(t) = t^2}$

c) Euler with $h = 1/2$ $\quad y_0 = 1 \quad t_0 = 1$, $t_1 = 3/2$, $t_2 = 2$

$f(t,y) = \frac{2y^2}{t^3}$

$$y_1 = y_0 + h\, f(t_0, y_0)$$

$$= 1 + \tfrac{1}{2}\left(\frac{2\cdot 1}{1^3}\right) = 1+1 = 2$$

$$y_2 = y_1 + h\, f(t_1, y_1)$$

$$= 2 + \tfrac{1}{2}\left(\frac{2\cdot 2^2}{(3/2)^3}\right)$$

$$= 2 + \frac{8}{2}\left(\frac{2}{3}\right)^3 = 2 + \frac{32}{27} = \boxed{\frac{86}{27}}$$

(exact solution is $2^2 = 4$... we missed it a bit

5) Notes

1) equilibrium at 0, −2
2) no equilibrium, ~~un~~slopes horizontal if $y=t$
3) equilibrium $y=-1$
4) equilibrium $y=-2$ vertical slope at $y=0$
semistable

a)

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A)	4
B)	1
C)	3
D)	2

1+ explanation

1	B
2	D
3	C
4	A

b)

7

1) nonlinear, autonomous
2) nonlinear, nonautonomous
3) linear, nonautonomous
4) nonlinear autonomous

1 pt each

c) 4 A) semistable equilibrium at −2 (> on both sides)

5 1 B) ~~no equilibria~~ unstable at 0, semistable at −2

3 C) unstable at $y=-1$

2 D) no equilibria