1

a) $y' = \cos t\, e^{\sin t - y} = \cos t\, e^{\sin t} e^{-y} = \frac{dy}{dt}$

$\Rightarrow e^{y} dy = e^{\sin t} \cos t\, dt$ integrating,

$e^{y} = \int e^{\sin t} \cos t\, dt = \int e^{u} du = e^{u} + C = e^{\sin t} + C$

$u = \sin t$
$du = \cos t\, dt$

$$\boxed{y = \ln | e^{\sin t} + C|}$$

$y(0) = \ln | e^{0} + C| = \ln |1 + C| = 1 \Rightarrow 1 + C = e$

$$\boxed{C = e - 1}$$

2:36

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b) $ty' + y = \frac{t}{t^2+1} \xrightarrow[\text{form}]{\text{normal}} y' + \frac{1}{t} y = \frac{1}{t^2+1}$

$(uy)' = uy' + u'y = uy' + u\frac{1}{t}y = \frac{u}{t^2+1}$

$\Downarrow$

$u' = \frac{u}{t} \Rightarrow \frac{du}{u} = \frac{dt}{t} \Rightarrow u = t$

$\Rightarrow (ty)' = \frac{t}{t^2+1}$ integrating, $ty = \frac{1}{2}\int \frac{2t\, dt}{t^2+1} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(1+t^2) + C$

$u = t^2+1$
$du = 2t\, dt$

$$\boxed{y = \frac{1}{2t}\ln(1+t^2) + \frac{C}{t}}$$

5:19.

② $y'' - 2y' + y = \frac{e^t}{t}$

a) $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0 \Rightarrow r = \frac{2 \pm \sqrt{4-4}}{2} = 1$ (repeated)

6:54

$y_h = c_1 e^t + c_2 t e^t$

b) Variation of parameters: $y_p = V_1 e^t + V_2 t e^t$

$V_1' e^t + V_2' t e^t = 0 \quad \times \frac{1}{e^t} \Rightarrow \quad V_1' + V_2' t = 0 \quad (1)$

$V_1' e^t + V_2'(te^t + e^t) = \frac{e^t}{t} \qquad V_1' + V_2'(t+1) = \frac{1}{t} \quad (2)$

(1) − (2): $-V_2' = -\frac{1}{t} \qquad V_2' = \frac{1}{t} \quad \overset{\text{in (1)}}{\Rightarrow} \quad V_1' = -1$

$\Rightarrow V_1 = -t \qquad V_2 = \ln t$

$y_p = -te^t + (\ln t) t e^t$

10:10

c) $y = y_h + y_p = c_1 e^t + c_2 t e^t - t e^t + (\ln t) t e^t$

10:42

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③ $x' = x + y$
$y' = y - x$

$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

eigenvalues:
$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = 0$

$1 - \lambda = \pm i \qquad \lambda = 1 \pm i$

Eigenvector for $\lambda = 1 - i$

$(A - \lambda I)\vec{v} = 0$: $\left[\begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array}\right] \xrightarrow{①\cdot i + ② \to ②} \left[\begin{array}{cc|c} i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad iv_1 = -v_2 \quad \vec{v} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \to \vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$y_1 = e^{(1-i)t} \begin{bmatrix} i \\ 1 \end{bmatrix} = e^t(\cos t - i \sin t)\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

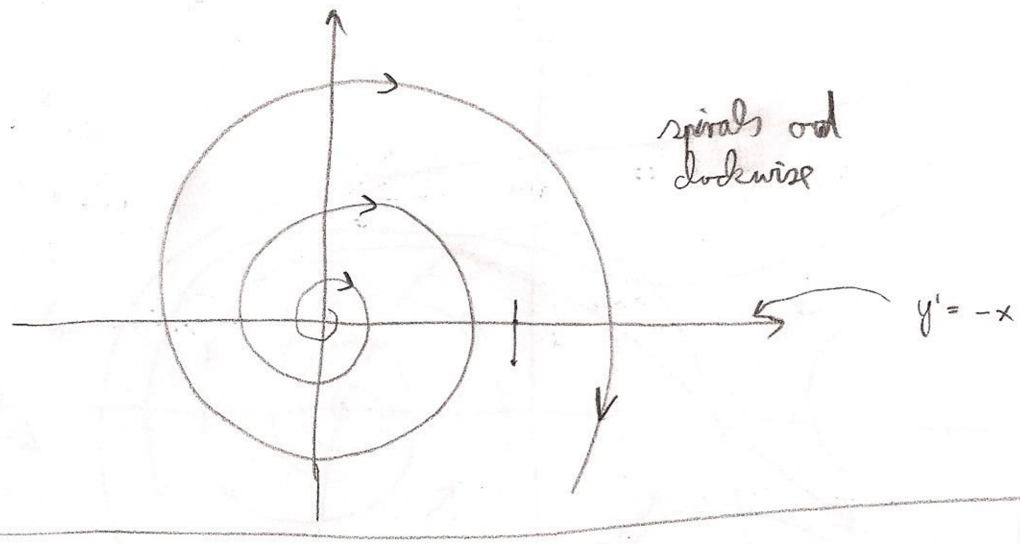
$= e^t\left(\cos t \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + i\, e^t\left(\cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

(3) continued

$$\Rightarrow \vec{y}(t) = C_1 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + C_2 e^t \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

16:06

b) $\alpha > 0$



17:11

(4)

$$x' = y$$
$$y' = 1 - x^2 - y$$

a) Fixed points: $y = 0$, $1 - x^2 - y = 0$ $\Rightarrow$ $y = 0$, $x^2 = 1 \Rightarrow x = \pm 1$ $\Rightarrow$ F. P. $(1,0)$, $(-1,0)$

Jacobian: $J = \begin{bmatrix} 0 & 1 \\ -2x & -1 \end{bmatrix}$

@ $(1,0)$ $J = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$

@ $(-1,1)$ $J = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}$

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b) @$(1,0)$ $|J - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -2 & -1-\lambda \end{vmatrix} = \lambda(1+\lambda) + 2 = \lambda^2 + \lambda + 2 = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1-8}}{2}$

(asympt. stable) <u>spiral in</u>

@ $(-1,0)$ $|J - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 2 & -1-\lambda \end{vmatrix} = \lambda(\lambda+1) - 2 = \lambda^2 + \lambda - 2 = 0 \Rightarrow \lambda = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2}$

<u>saddle</u>

(unstable)

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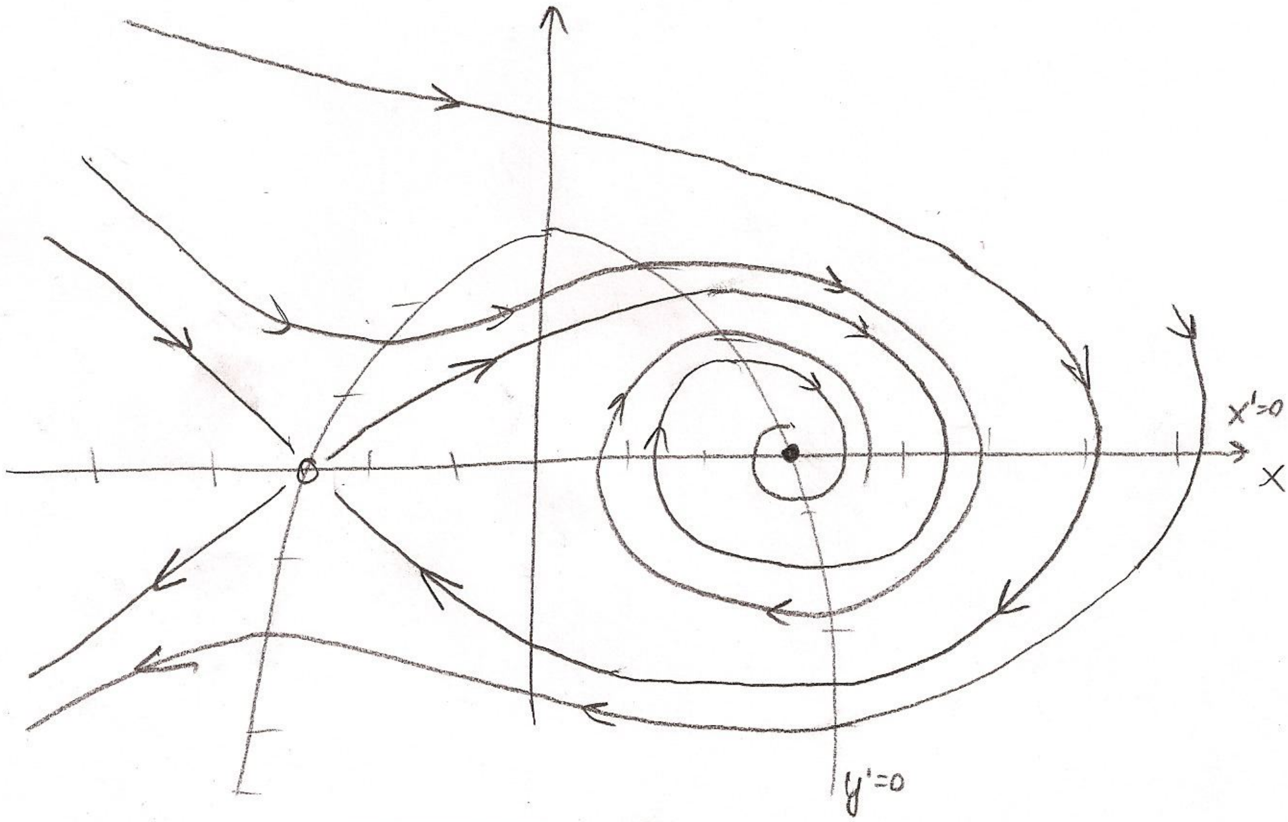
4 c) Nullclines $y'=0 \Rightarrow y = 1-x^2$

$x'=0 \Rightarrow y=0$



⑤ $A\vec{x}=\vec{b}$ $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

a)

$$|A| = 1\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = (1-2) - 0 = -1$$

b) $|A| \neq 0$ so there is a unique solution

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ -1 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_1+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 2 & 2 \end{array}\right] \xrightarrow{R_3-3R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right] \xrightarrow[\substack{R_3+R_1 \to R_1 \\ -R_3 \to R_3}]{R_3+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1-R_2 \to R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{l} x=1 \\ y=0 \\ z=1 \end{array}$$

Check: $x+y+z = 1+0+1=2$ ✓

$y+z = 0+1 = 1$ ✓

$-x+2y+z = -1+0+1=0$ ✓

c) Since A is invertible, its columns are linearly independent

$\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}$ are the columns of A, therefore they are linearly independent

7)

a) $\frac{dh}{dt} = 0.1h \qquad \frac{ds}{dt} = -0.2s \Rightarrow h(t) = h(0)e^{0.1t}, \ s(t) = s(0)e^{-0.2t}$

b) $s(0) = 2h(0)$

$s(T) = h(T) \Longrightarrow s(0)e^{-0.2T} = h(0)e^{0.1T} = 2h(0)e^{-0.2T}$

$\Rightarrow e^{0.1T} = 2e^{-0.2T}$ taking $\ln(\ )$

$0.1T = \ln 2 - 0.2T \Rightarrow 0.3T = \ln 2 \Rightarrow \boxed{T = \frac{\ln 2}{0.3}}$

c) $\frac{dh}{dt} = 0.1L(t)h = 0.1(1 + \sin(2\pi t))h$ separating variables

$\frac{dh}{h} = 0.1(1+\sin(2\pi t))dt$ integrating $\ln|h| = 0.1t - \frac{0.1}{2\pi}\cos(2\pi t)$

$\Rightarrow h(t) = h(0)e^{0.1t - \frac{0.1}{2\pi}\cos(2\pi t)}$ as $t \to \infty$, $h(t) \longrightarrow \infty$

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6) $\ddot{x} - 2\dot{x} + 5x = f(t)$

a) $r^2 - 2r + 5 = 0 \Rightarrow r = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$

$x_h = c_1 e^t \cos 2t + c_2 e^t \sin 2t$

b) $\boxed{x_p = A + Bt + Ct^2}$

$\dot{x}_p = B + 2Ct \qquad \Rightarrow 2C - 2B - 4Ct + 5A + 5Bt + 5Ct^2 = 25t^2$

$\ddot{x}_p = 2C \qquad \boxed{C = 5} \qquad 2C - 2B + 5A = 0 \qquad 5A = 8 - 10 = -2$

$-4C + 5B = 0 \Rightarrow \boxed{B = 4} \qquad \boxed{A = \frac{-2}{5}}$

41:00

7 c) $X_p(t) = (A+Bt)\sin(2t) + (C+Dt)\cos(2t)$

(8)

a) FALSE

b) TRUE

c) TRUE $\vec{x}=0$

d) TRUE

e) FALSE

f) TRUE

g) TRUE

h) TRUE