

**APPM 2360: Section exam 2**  
7.00pm – 8.30pm, February 11, 2009.

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ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

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**Problem 1:** (36 points) Give a brief answer to each question. Box your answer. Each correct answer earns 4 points. No work given for this question will be graded.

- (a) Let  $\mathbf{v}_1$ ,  $\mathbf{v}_2$ , and  $\mathbf{v}_3$  be non-zero vectors in  $\mathbb{R}^7$ . Let  $r$  denote the dimension of  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ . What are the possible values of  $r$ ?

(b) Set  $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ ,  $C = \begin{bmatrix} 2 & 3 & 2 \\ 1 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ , and  $X = ABC$ .

Compute  $\det(X)$ . *Hint:*  $\det(A) = -3$  and  $\det(C) = -2$ .

- (c) Which of the following sets constitute basis sets for  $\mathbb{P}_2$  (the set of all polynomials of degree at most 2):

$$S_1 = \{1, t, t^2\}, \quad S_2 = \{1, t^2\}, \quad S_3 = \{1 + t, 1 - t, t^2\}, \quad S_4 = \{0, 1, t, t^2\}.$$

- (d) Assuming the standard addition and multiplication rules, determine which of the following sets are vector spaces.

(i) All  $4 \times 5$  matrices  $A$  such that  $a_{2,3} = 0$ ;

(ii) All vectors  $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$  that satisfy  $x_1 + x_2 = 3$ .

(e) Set  $A = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix}$ ,  $B = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ . Find  $A^T B$ .

- (f) Let  $A$  be an  $8 \times 5$  matrix. If the rank of  $A$  is 2, what is the dimension of the vector space  $V = \{\mathbf{x} \in \mathbb{R}^5 : A\mathbf{x} = 0\}$ .

(g) For which values of  $t$  is the matrix  $A = \begin{bmatrix} 1 & t \\ 1 & (1+t) \end{bmatrix}$  invertible?

- (h) "If a  $4 \times 4$  matrix  $A$  is invertible, then  $A^5$  is also invertible." Answer TRUE or FALSE.

- (i) "The dimension of the vector space of all  $m \times n$  matrices is  $m+n$ ." Answer TRUE or FALSE.

For question 2 — 5, motivate your answers. A correct answer with no work may receive no credit, while an incorrect answer with some correct work may result in partial credit.

**Problem 2:** (16 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

Let  $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  denote the columns of  $A$ , so that  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ ,  $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ,  $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ .

(a) (8 points) Is  $A$  invertible? If yes, find the inverse; if not, substantiate your claim.

(b) (4 points) Is the set  $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  linearly independent? Justify your answer.

(c) (4 points) Determine for which real numbers  $t$  (if any), the vector  $\mathbf{b} = \begin{bmatrix} 1 \\ t \\ 1 \end{bmatrix}$  belongs to  $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ .

*Hint:* Verify your calculations in (a) carefully since a wrong answer may influence your answers to (b) and (c).

**Problem 3:** (16 points) Set

$$A = \begin{bmatrix} 1 & -2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 & 0 \\ -1 & 0 & -2 & 1 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 3 \\ -2 \\ 3 \end{bmatrix}.$$

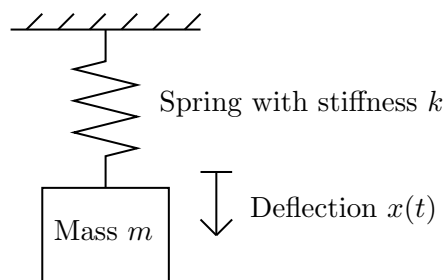
(a) (8 points) Give the general solution of the equation  $A\mathbf{x} = \mathbf{b}$ .

(b) (8 points) Construct a basis for the vector space  $V = \text{Null}(A) = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$ .

*Hint:*  $A$  is row equivalent to the matrix  $\begin{bmatrix} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ .

**Problem 4:** (16 points) Consider the mechanical spring-mass system shown in the picture. Let  $x = x(t)$  denote the deflection of the spring from its equilibrium position (counted positive downwards) at time  $t$ . The governing equation is

$$\ddot{x} + 9x = 0. \quad (1)$$



- (a) (4 points) Suppose that the body has mass  $m$  and the spring has stiffness  $k$ . What equation must  $m$  and  $k$  satisfy for the system to be modeled by (1)?
- (b) (4 points) Give the general solution of (1).
- (c) (8 points) Suppose that at time  $t = 1$ , the mass is at position  $x = 0$  and has (downwards) velocity  $v_0$ . Determine the solution  $x(t)$  of (1) that matches these conditions. (Your answer should depend on the parameter  $v_0$ .)

**Problem 5:** (16 points) Let  $k$  be a real number and consider the system

$$\begin{aligned} (1 - k)x + (1 - k)y &= 1, \\ ky &= k. \end{aligned}$$

- (a) (4 points) Find all the values of  $k$ , if any, for which this system has infinitely many solutions.
- (b) (4 points) Find all the values of  $k$ , if any, for which this system has no solutions.
- (c) (8 points) Find all the values of  $k$ , if any, for which this system has a unique solution. Determine  $x$  and  $y$  in this case.