Problem 1

a.) yes
b.) No
c.) No
d.) No
e.) yes

Problem 2

a.) $LHS = Tr(A+B) = \sum_{k=1}^{n} (a_{kk} + b_{kk}) = \sum_{k=1}^{n} a_{kk} + \sum_{k=1}^{n} b_{kk} = TrA + TrB$
$= RHS$

b.) $LHS = Tr(cA) = \sum_{k=1}^{n} (ca_{kk}) = c \sum_{k=1}^{n} a_{kk} = c\,Tr(A) = RHS$

c.) we have to show that W is closed under addition and scalar multiplication.

Let $\vec{u}$ and $\vec{v}$ be in W

so $\vec{u} = \begin{bmatrix} u_1 & 0 \\ 0 & -u_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 & 0 \\ 0 & -v_1 \end{bmatrix}$

$\Rightarrow c_1\vec{u} + c_2\vec{v} = \begin{bmatrix} c_1u_1 + c_2v_1 & 0 \\ 0 & -c_1u_1 - c_2v_2 \end{bmatrix}$

$= \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} \in W$

Let $c_1u_1 + c_2v_2 = a$

So W is closed under addition and scalar mult.
so W is a subspace of M_{22}

d.) basis $= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$ and $\dim(W) = 1$

Problem 3

a) $\left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 1 & 1 & 1 & b \\ 1 & 1 & 1 & b \end{array}\right] \xrightarrow[R_3 - R_1 \to R_3]{R_2 - R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

b) 1 pivot, rank$(A) = 1$, infinitely many solutions

c) Let $x_2 = s \in \mathbb{R}$ and $x_3 = t \in \mathbb{R}$

$x_1 + x_2 + x_3 = b \Rightarrow x_1 = b - s - t$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b-s-t \\ s \\ t \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

<u>Not a vector Space</u> since $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in the solution set.

Problem 4

$A = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

a.) $\left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right] \xrightarrow{-2R_2 + R_1 \to R_2} \left[\begin{array}{cc|cc} 2 & 0 & 1 & 0 \\ 0 & -2 & 1 & -2 \end{array}\right]$

$\xrightarrow[-\frac{1}{2}R_2]{\frac{1}{2}R_1} \left[\begin{array}{cc|cc} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/2 & 1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$

b.) $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1}\vec{b} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

c.) $|A| = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 - 0 = 2$

$|A_1| = \left|\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}\right| = 2 - 0 = 2 \Rightarrow x_1 = \frac{|A_1|}{|A|} = \frac{2}{2} = 1$

$|A_2| = \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} = 4 - 2 = 2 \Rightarrow x_2 = \frac{|A_2|}{|A|} = \frac{2}{2} = 1$

$\Rightarrow \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Problem 5

a.) $A\vec{v} = \lambda \vec{v}$

b.) $A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & -3 & -1 \\ 0 & 2 & -1 \end{bmatrix}$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & -1 \\ 0 & -3-\lambda & -1 \\ 0 & 2 & -1-\lambda \end{vmatrix}$$

$$= (3-\lambda) \begin{vmatrix} -3-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} = (3-\lambda)\left[(-3-\lambda)(-1-\lambda) + 2\right]$$

$$= (3-\lambda)(3 + 4\lambda + \lambda^2 + 2)$$

$$= (3-\lambda)(\lambda^2 + 4\lambda + 5)$$

$$\Rightarrow \boxed{\lambda_1 = 3} \qquad \lambda_{2,3} = \frac{-4 \pm \sqrt{16-20}}{2} = \boxed{-2 \pm i = \lambda_{2,3}}$$

c.)

$$\lambda_1 = 3 \Rightarrow \left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & -6 & -1 & 0 \\ 0 & 2 & -4 & 0 \end{array}\right] \begin{array}{l} \rightarrow x_3 = 0 \\ \rightarrow x_1 = r \in \mathbb{R} \\ \rightarrow x_2 = 0 \end{array} \Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$$

$$\lambda_2 = -2 + i \Rightarrow \left[\begin{array}{ccc|c} 5-i & 0 & -1 & 0 \\ 0 & -1-i & -1 & 0 \\ 0 & 2 & 1-i & 0 \end{array}\right] \xrightarrow[(-1+i)R_2]{(5+i)R_1} \left[\begin{array}{ccc|c} 26 & 0 & -5-i & 0 \\ 0 & 2 & 1-i & 0 \\ 0 & 2 & 1-i & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2 \to R_3} \left[\begin{array}{ccc|c} 26 & 0 & -5-i & 0 \\ 0 & 2 & 1-i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \rightarrow 26x_1 - (5+i)x_3 = 0 \\ \rightarrow 2x_2 + (1-i)x_3 = 0 \\ \rightarrow x_3 = r \in \mathbb{R} \end{array}$$

$$\Rightarrow x_1 = \frac{5+i}{26} r \quad \text{and} \quad x_2 = \frac{(-1+i)}{2} r$$

if $r = 26$

$$\Rightarrow \vec{V}_2 = \begin{bmatrix} 5+i \\ 13(-1+i) \\ 26 \end{bmatrix} = \begin{bmatrix} 5 \\ -13 \\ 26 \end{bmatrix} + i \begin{bmatrix} 1 \\ 13 \\ 0 \end{bmatrix} \Rightarrow V_3 = \begin{bmatrix} 5 \\ -13 \\ 26 \end{bmatrix} - i \begin{bmatrix} 1 \\ 13 \\ 0 \end{bmatrix}$$