APPM 2360 Final Summer 2009

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) your lecture section, (4) your instructor's name and (5) a grading table. You have 90 minutes to work all 5 problems on the exam. The point values are indicated at the start of each problem; there are 170 points total. Show ALL of your work in the bluebook and box in final answers. Start each problem on a new page. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Text books, class notes and calculators are NOT permitted. One letter size (8.5″ × 11″) crib sheet with anything hand written on both sides is allowed.

1. (15 points) Answer each of the following True/False questions. You do not need to justify your answer.

   (a) **True or False** The differential equation $y' = \dfrac{y}{y-t}$ has a unique solution for the initial condition $y(1) = 1$

   (b) **True or False** In the method of Undetermined Coefficients the predicted form of the particular solution for the differential equation $y'' - 6y + 9y = e^{3t}$ is $y_p(t) = (At + B)e^{3t}$.

   (c) **True or False** The chaotic motion in the Lorenz system is not sensitive to initial conditions.

2. (40 points) Consider the differential equation

$$\frac{dy}{dt} = y^2 - 4$$

   (a) Find the solution to the differential equation with initial condition $y(0) = 0$.

   (b) Find the equilibrium solutions and determine the stability of each one.

   (c) Draw the direction field and sketch some sample solutions, including the equilibrim solutions and the solution found in part (a). (**Note:** This plot will be in the (t,y)-plane).

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3. (40 points) Consider the following 2nd order initial value problem.

$$y'' - y' - 2y = 10 \sin t \qquad\qquad y(0) = 1, \quad y'(0) = 0$$

(a) Find the solution $y_h(t)$ to the corresponding homogeneous differential equation.

(b) Find the particular solution $y_p(t)$ to the non-homogeneous differential equation using either **method of undetermined coefficients** or **variation of parameters**.

(c) Use the given initial conditions to solve the IVP.

(d) Convert the **homogeneous** equation to a system of two first order equations. Include the initial conditions in the conversion.

(e) Solve the converted **homogeneous** IVP using usual techniques for a 2-dimensional, first-order linear system with constant coefficients.

(f) Sketch the phase-portait for the 2-dimensional system.

4. (35 points) Fun with Linear Algebra
Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix}$$

(a) What is the rank of A? Is A invertible?

(b) How many solutions are there for the linear system

$$A\vec{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

(c) Find the eigenvalues, determinant and trace of A.

5. (40 points) Consider the following equation, known as the Lotka-Volterra predator-prey system. We looked at this early in chapter 2, but now have more tools to analyze the behavior of this 2-dimensional, first-order, nonlinear system of differential equations.

$$x' = (a - by)x$$
$$y' = (cx - d)y$$

(a) Find the equilibrium points of this system. (Note: At least one of your answers will be in terms of a, b, c and d.)

(b) Now use the parameter values $a = 2$, $b = 1$, $c = \frac{1}{2}$ and $d = 3$. Write down the equilibrium points for these parameter values. Analyze the stability behavior of the system locally, around each of these points.

(c) For the same parameter values as in part (b), draw the phase portrait of the system, include all fixed points and illustrate the behavior of the system about these points.