Problem 1

a.) False

b.) False

c.) False

Problem 2

a.) $\int \frac{dy}{y^2-4} = \int dt$

$$\frac{1}{y^2-4} = \frac{A}{y+2} + \frac{B}{y-2}$$

$$1 = A(y-2) + B(y+2)$$

if $y=2 \Rightarrow B = \frac{1}{4}$

if $y=-2 \Rightarrow A = \frac{-1}{4}$

$$\frac{1}{4}\int\left(\frac{-1}{y+2} + \frac{1}{y-2}\right)dy = \int dt$$

$$\Rightarrow \frac{1}{4}\left(-\ln|y+2| + \ln|y-2|\right) = t + C$$

$$\Rightarrow \ln\left|\frac{y-2}{y+2}\right| = 4t + C$$

$y(0)=0 \Rightarrow \ln|-1| = 0 + C \Rightarrow C=0$

$$\Rightarrow \left|\frac{y-2}{y+2}\right| = e^{4t}$$

Choose negative because of initial condition

$$\Rightarrow 2-y = e^{4t}(y+2)$$

$$\Rightarrow y(-e^{4t}-1) = 2e^{4t}-2$$

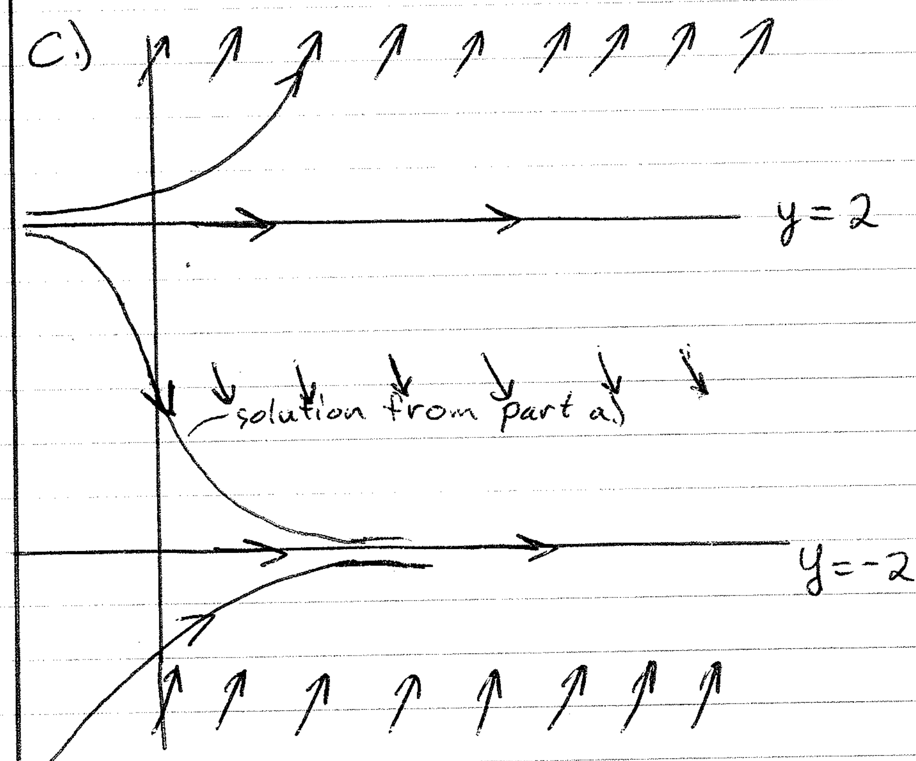
$$\Rightarrow \boxed{y(t) = \frac{2(1-e^{4t})}{e^{4t}+1}}$$

Prob 2 b) Equilibria: $y' = 0 = y^2 - 4$

$\Rightarrow y = \pm 2$

y	y'	
-3	$5 > 0$	
-2	0	$\longrightarrow$ $y = -2$ is stable
0	$-4 < 0$	
2	0	$\longrightarrow$ $y = 2$ is unstable
3	$5 > 0$	

c.)



Problem 3

$$y'' - y' - 2y = 10\sin t \qquad , \; y(0) = 1 \; , \; y(0) = 0$$

a.) $r^2 - r - 2 = 0$

$(r-2)(r+1) = 0 \Rightarrow r_1 = 2, \; r_2 = -1$

$\Rightarrow y_h(t) = c_1 e^{2t} + c_2 e^{-t}$

b.) $y_p(t) = A\cos t + B\sin t$

$y_p'(t) = -A\sin t + B\cos t$

$y_p''(t) = -A\cos t - B\sin t$

$\Rightarrow y_p'' - y_p' - 2y_p = -A\cos t - B\sin t + A\sin t - B\cos t - 2A\cos t - 2B\sin t$

$= 10\sin t$

$\Rightarrow -3A - B = 0 \rightarrow B = -3A$

$A - 3B = 10 \rightarrow A + 9A = 10$

$\rightarrow A = 1 \Rightarrow B = -3$

$\Rightarrow \boxed{y_p(t) = \cos t - 3\sin t}$

c.) $y(t) = c_1 e^{2t} + c_2 e^{-t} + \cos t - 3\sin t$

$y'(t) = 2c_1 e^{2t} - c_2 e^{-t} - \sin t - 3\cos t$

Initial Cond's: $y(0) = 1 = c_1 + c_2 + 1 \Rightarrow c_1 = -c_2$

$y'(0) = 0 = 2c_1 - c_2 - 3 \Rightarrow 3 = 3c_1$

$\Rightarrow c_1 = 1, \; c_2 = -1$

$\Rightarrow \boxed{y(t) = e^{2t} - e^{-t} + \cos t - 3\sin t}$

Prob 3 d)

$$y_1 = y$$
$$y_2 = y' = y_1'$$
$$y_2' = y' + 2y = y_2 + 2y_1$$

$$\Rightarrow \boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= 2y_1 + y_2 \end{aligned}}$$

$$\boxed{y_1(0) = 1,\ y_2(0) = 0}$$

e)
$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$$

evals: $\begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = -\lambda + \lambda^2 - 2 = 0$

$$\Rightarrow \lambda_1 = 2,\ \lambda_2 = -1$$

$\lambda_1 = 2$ $\left[\begin{array}{cc|c} -2 & 1 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to -2\mu_1 + \mu_2 = 0$

$\mu_1 = s,\ \mu_2 = 2s$

$$\to \vec{V}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\lambda_2 = -1$ $\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 2 & 0 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \mu_1 + \mu_2 = 0$

$\mu_2 = s,\ \mu_1 = -s$

$$\vec{V}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

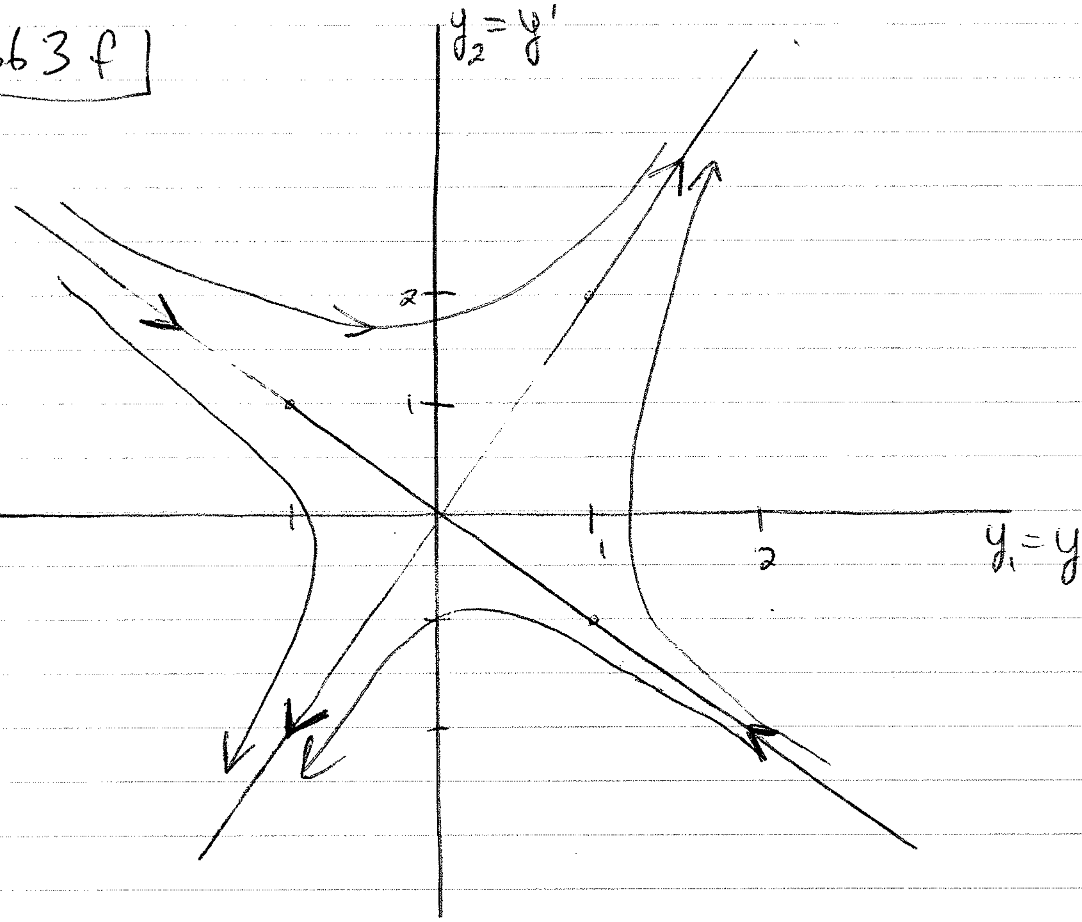
$$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1 e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Init. cond: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 0 \end{array}\right] \xrightarrow{-2R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & -2 \end{array}\right] \xrightarrow{3R_1 + R_2 \to R_1} \left[\begin{array}{cc|c} 3 & 0 & 1 \\ 0 & 3 & -2 \end{array}\right]$$

$$\to c_1 = \tfrac{1}{3},\ c_2 = -\tfrac{2}{3} \to \boxed{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \tfrac{1}{3} e^{2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \tfrac{2}{3} e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

Prob 3 f



Problem 4

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{bmatrix}$$

a) $\begin{vmatrix} 1 & 0 & 2 \\ -1 & 1 & -1 \\ 1 & 1 & 3 \end{vmatrix} = 3-2-(2-1) = 1-1 = 0$

A is not invertible

rank$(A) = 2$

b) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ -1 & 1 & -1 & 3 \\ 1 & 1 & 3 & 1 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & 1 & -1 \end{array}\right]$

$\xrightarrow{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & -6 \end{array}\right] \longrightarrow$ No Solution

$\longrightarrow$ 2 pivot columns so rank$(A) = 2$

c.) $\boxed{|A| = 0}$ from part a.)

$\begin{vmatrix} 1-\lambda & 0 & 2 \\ -1 & 1-\lambda & -1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)(3-\lambda)+1\right] + 2(-1-1+\lambda)$

$= (1-\lambda)(4-4\lambda+\lambda^2) - 4 + 2\lambda$

$= 4-4\lambda+\lambda^2-4\lambda+4\lambda^2-\lambda^3-4+2\lambda$

$= -\lambda^3+5\lambda^2-6\lambda = 0 \Rightarrow \boxed{\lambda_1 = 0,\ \lambda_2 = 3,\ \lambda_3 = 2}$

$= -\lambda(\lambda^2-5\lambda+6) = 0$

$\boxed{\mathrm{Tr}A = 1+1+3 = 5}$

Problem 5

a) v-clines: $y = \frac{a}{b}$, $x = 0$
h-clines: $x = \frac{d}{c}$, $y = 0$

equilibria: $(x, y) = (0, 0)$ and $(\frac{d}{c}, \frac{a}{b})$

b) $a = 2$, $b = 1$, $c = \frac{1}{2}$, $d = 3$

equilibria: $(0,0)$, $(6, 2)$

$$J(x_e, y_e) = \begin{bmatrix} a - by_e & -bx_e \\ cy_e & cx_e - d \end{bmatrix} = \begin{bmatrix} 2 - y_e & -x_e \\ \frac{1}{2} y_e & \frac{1}{2}x_e - 3 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \Rightarrow \lambda_1 = 2, \lambda_2 = -3$$

Thus $(0,0)$ is an unstable saddle equilibrium

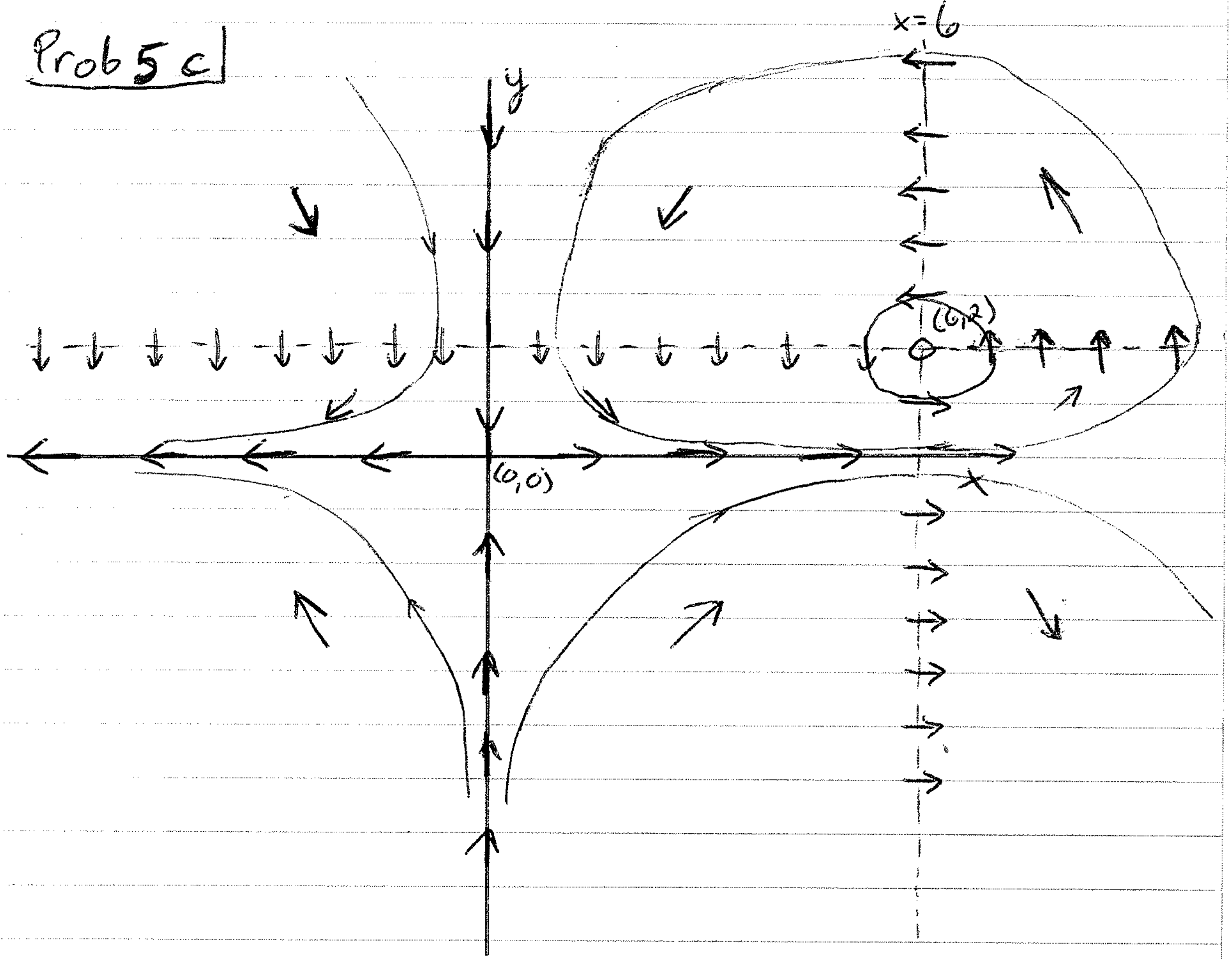
$$J(6,2) = \begin{bmatrix} 0 & -6 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{vmatrix} -\lambda & -6 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 6 = 0$$

$$\lambda_{1,2} = \pm\sqrt{6}\, i$$

Thus $(6,2)$ is a center or spiral and stability is uncertain.

$$x' = (2-y)x$$
$$y' = \left(\tfrac{1}{2}x - 3\right)y$$

Prob 5 c



Take Home Problem 1

$$t y'' + y' + \left(t - \frac{1}{4t}\right)y = \sqrt{t}$$

a.) $y_1 = t^{-\frac{1}{2}} \sin t$

$$y_1' = -\tfrac{1}{2} t^{-3/2} \sin t + t^{-\frac{1}{2}} \cos t$$

$$y_1'' = \tfrac{3}{4} t^{-5/2} \sin t - \tfrac{1}{2} t^{-3/2} \cos t - \tfrac{1}{2} t^{-\frac{1}{2}} \cos t - t^{-\frac{1}{2}} \sin t$$

$$= \tfrac{3}{4} t^{-5/2} \sin t - t^{-3/2} \cos t - t^{-\frac{1}{2}} \sin t$$

$$t y_1'' + y_1' + \left(t - \frac{1}{4t}\right) y_1 = \cancel{\tfrac{3}{4} t^{-3/2} \sin t} - \cancel{t^{-\frac{1}{2}} \cos t} - \cancel{t^{\frac{1}{2}} \sin t}$$
$$- \cancel{\tfrac{1}{2} t^{-3/2} \sin t} + \cancel{t^{-\frac{1}{2}} \cos t}$$
$$+ \cancel{t^{1/2} \sin t} - \cancel{\tfrac{1}{4} t^{-3/2} \sin t}$$
$$= 0$$

$y_2 = t^{-\frac{1}{2}} \cos t$

$$y_2' = -\tfrac{1}{2} t^{-3/2} \cos t - t^{-\frac{1}{2}} \sin t$$

$$y_2'' = \tfrac{3}{4} t^{-5/2} \cos t + \tfrac{1}{2} t^{-3/2} \sin t + \tfrac{1}{2} t^{-\frac{3}{2}} \sin t - t^{-\frac{1}{2}} \cos t$$

$$t y_2'' + y_2' + \left(t - \frac{1}{4t}\right) y_2 = \cancel{\tfrac{3}{4} t^{-3/2} \cos t} + \cancel{t^{-\frac{1}{2}} \sin t} - \cancel{t^{\frac{1}{2}} \cos t}$$
$$- \cancel{\tfrac{1}{2} t^{-3/2} \cos t} - \cancel{t^{-\frac{1}{2}} \sin t}$$
$$+ \cancel{t^{\frac{1}{2}} \cos t} - \cancel{\tfrac{1}{4} t^{-3/2} \cos t}$$
$$= 0$$

b.) $$W[y_1, y_2] = \begin{vmatrix} t^{-\frac{1}{2}} \sin t & t^{-\frac{1}{2}} \cos t \\ -\frac{1}{2} t^{-3/2} \sin t + t^{-\frac{1}{2}} \cos t & -\frac{1}{2} t^{-3/2} \cos t - t^{-\frac{1}{2}} \sin t \end{vmatrix}$$

$$= \cancel{-\tfrac{1}{2} t^{-2} \sin t \cos t} - t^{-1} \sin^2 t + \cancel{\tfrac{1}{2} t^{-2} \sin t \cos t} - t^{-1} \cos^2 t$$

$$= -t^{-1} = \frac{-1}{t} \neq 0 \implies y_1 \text{ and } y_2 \text{ are linearly independent.}$$

c.) $$y_h(t) = t^{-1/2}(c_1 \sin t + c_2 \cos t)$$

T.H. Prob 1d

$$y_1 v_1' + y_2 v_2' = 0$$
$$y_1' v_1' + y_2' v_2' = f = \frac{\sqrt{t}}{t} = t^{-1/2}$$

Cramer's Rule gives:

$$v_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-y_2 f}{W[y_1, y_2]}$$

$$v_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 f}{W[y_1, y_2]}$$

e.) $$v_1' = \frac{-t^{-1/2}\cos t \cdot t^{-1/2}}{-t^{-1}} = \cos t \Rightarrow \boxed{v_1 = \sin t}$$

$$v_2' = \frac{t^{-1/2}\sin t \cdot t^{-1/2}}{-t^{-1}} = -\sin t \Rightarrow \boxed{v_2 = \cos t}$$

f.) $$y_p(t) = t^{-1/2}\sin^2 t + t^{-1/2}\cos^2 t = t^{-1/2}$$

$$\boxed{y_p(t) = \frac{1}{\sqrt{t}}}$$

g.) $$\boxed{y(t) = t^{-1/2}\left[c_1 \sin t + c_2 \cos t + 1\right]}$$

T.H. Problem 2

a.) $U' = 1(0-U) + 2(W-U) + 2(E-U)$

$= -5U + 2W + 2E$

$W' = 1(0-W) + 2(U-W) + 3(E-W)$

$= 2U - 6W + 3E$

$E' = 1(0-E) + 2(U-E) + 3(W-E)$

$= 2U + 3W - 6E$

$U(0) = 60\,°F$, $W(0) = 70°F$, $E(0) = 80°F$

b.) $$\begin{bmatrix} U \\ W \\ E \end{bmatrix}' = \begin{bmatrix} -5 & 2 & 2 \\ 2 & -6 & 3 \\ 2 & 3 & -6 \end{bmatrix} \begin{bmatrix} U \\ W \\ E \end{bmatrix}$$

$$\begin{bmatrix} U(0) \\ W(0) \\ E(0) \end{bmatrix} = \begin{bmatrix} 60 \\ 70 \\ 80 \end{bmatrix}$$

evals: $$\begin{vmatrix} -5-\lambda & 2 & 2 \\ 2 & -6-\lambda & 3 \\ 2 & 3 & -6-\lambda \end{vmatrix} = (-5-\lambda)\left((-6-\lambda)^2-9\right) - 2(-12-2\lambda-6) + 2(6+12+2\lambda)$$

$= -(5+\lambda)(27+12\lambda+\lambda^2) + 72 + 8\lambda$

$= -135 - 60\lambda - 5\lambda^2 - 27\lambda - 12\lambda^2 - \lambda^3 + 72 + 8\lambda$

$= -(63 + 79\lambda + 17\lambda^2 + \lambda^3) = 0$

$$\begin{array}{r|rrrr} -1 & 1 & 17 & 79 & 63 \\ & & -1 & -16 & -63 \\ \hline & 1 & 16 & 63 & 0 \end{array}$$

$\boxed{\lambda_1 = -1}$

$\lambda^2 + 16\lambda + 63 = 0$

$(\lambda+9)(\lambda+7) = 0$

$\boxed{\lambda_2 = -7,\ \lambda_3 = -9}$

cont. 2b.)

evecs:

$$\lambda_1 = -1 \quad \left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -5 & 3 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 2 & -5 & 3 & 0 \\ 2 & 3 & -5 & 0 \end{array}\right]$$

$$\xrightarrow[R_1+R_3]{R_1+R_2} \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 4 & -4 & 0 \end{array}\right] \xrightarrow[R_2/4]{R_2+R_3} \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_2+R_1 \left[\begin{array}{ccc|c} -2 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \rightarrow \mu_1 = \mu_2 \\ \rightarrow u_2 = \mu_3 \end{array} \rightarrow \boxed{\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

$$\lambda_2 = -7 \quad \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 1 & 3 & 0 \\ 2 & 3 & 1 & 0 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3,\ R_1/2]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{r} R_2+R_1 \to R_1 \\ R_2+R_3 \to R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \rightarrow \mu_1 = -2\mu_3 \\ \rightarrow \mu_2 = \mu_3 \end{array} \rightarrow \boxed{\vec{V}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}$$

$$\lambda_3 = -9 \quad \left[\begin{array}{ccc|c} 4 & 2 & 2 & 0 \\ 2 & 3 & 3 & 0 \\ 2 & 3 & 3 & 0 \end{array}\right] \xrightarrow[-R_2+R_3 \to R_3,\ -R_1+R_2 \to R_2]{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow[-R_2+R_1]{\frac{1}{2}R_2} \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} \mu_1 = 0 \\ \mu_2 = -\mu_3 \end{array} \rightarrow \boxed{\vec{V}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}}$$

cont. 2c)

$$\begin{bmatrix} U \\ W \\ E \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-7t}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{-9t}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

init. cond: $\begin{bmatrix} 60 \\ 70 \\ 80 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 60 \\ 1 & 1 & -1 & 70 \\ 1 & 1 & 1 & 80 \end{array}\right] \xrightarrow[-R_1+R_3 \to R_3]{-R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 60 \\ 0 & 3 & -1 & 10 \\ 0 & 3 & 1 & 20 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3 \to R_3]{2R_2+3R_1 \to R_1} \left[\begin{array}{ccc|c} 3 & 0 & -2 & 200 \\ 0 & 3 & -1 & 10 \\ 0 & 0 & 2 & 10 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c} 3 & 0 & -2 & 200 \\ 0 & 3 & -1 & 10 \\ 0 & 0 & 1 & 5 \end{array}\right]$$

$$\xrightarrow[2R_3+R_1 \to R_1]{R_3+R_2 \to R_2} \left[\begin{array}{ccc|c} 3 & 0 & 0 & 210 \\ 0 & 3 & 0 & 15 \\ 0 & 0 & 1 & 5 \end{array}\right] \to c_1 = 70,\ c_2 = 5,\ c_3 = 5$$

$$\to \boxed{\begin{aligned} U(t) &= 70e^{-t} - 10e^{-7t} \\ W(t) &= 70e^{-t} + 5e^{-7t} - 5e^{-9t} \\ E(t) &= 70e^{-t} + 5e^{-7t} + 5e^{-9t} \end{aligned}}$$

d.) $U(1) = 70e^{-1} - 10e^{-7} \approx 25.742°F$

$W(1) = 70e^{-1} + 5e^{-7} - 5e^{-9} \approx 25.7555°F$

$E(1) = 70e^{-1} + 5e^{-7} + 5e^{-9} \approx 25.757°F$

The insulation in the home is horrible since the temperature is dropping so fast