

Final Review Problems
APPM 2360
Fall 2009

1. (a) Solve the IVP

$$y' = \cos(t)e^{\sin(t)-y}$$
$$y(0) = 1$$

- (b) Find the general solution of

$$y' + \frac{1}{t}y = \frac{1}{t^2 + 1}.$$

2. Consider the matrix and the vector

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 2 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}$$
$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

- (a) For equation $\mathbf{Ax} = \mathbf{b}$

- i. Use Gauss-Jordan reduction to transform the augmented matrix of this system to RREF. *Show all your steps. Be sure to write down what row operations you are using.*
- ii. How many pivots does the matrix \mathbf{A} have? What's the rank of \mathbf{A} ? How many solutions does this linear system of equations have?
- iii. Find the general solution to the equation $\mathbf{Ax} = \mathbf{b}$.

- (b) Find the general solution to the equation $\mathbf{Ax} = \mathbf{0}$.

3. Consider the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Given that

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & 1 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

determine which of the following equations are solvable, and solve the ones that are.

(a) $\mathbf{Ax} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ with the 3×1 vector \mathbf{x} as unknown.

(b) $\mathbf{By} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ with the 3×1 vector \mathbf{y} as unknown.

(c) $\mathbf{ZA} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}$ with the 2×3 matrix \mathbf{Z} as unknown.

(d) $\mathbf{WA} + \mathbf{BWA} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with the 3×3 matrix \mathbf{W} as unknown.

4. Let \mathcal{V} be the set of all infinite sequence $\{s_n\}_{n=1}^{\infty} = \{s_1, s_2, \dots, s_n, \dots\}$, where the s_n 's are real numbers. Consider the subset \mathcal{W} consisting of those elements $\{s_n\}$ such that

$$s_{l+1} = s_l + s_{l-1}$$

for $l \geq 1$.

- (a) Show that \mathcal{W} is a subspace of \mathcal{V} .
(b) Show that \mathcal{W} is 2-dimensional.
5. Let PE_4 define the vector space of all even symmetric polynomials of degree ≤ 4 . All vector elements $v \in PE_4$ therefore have the form

$$v = a_0 + a_2t^2 + a_4t^4.$$

As an alternative interpretation, note that every vector $v \in PE_4$ can be thought of as a 3-vector $\mathbf{v} = (a_0, a_2, a_4)^T$.

- (a) What's the dimension of PE_4 ?
(b) Write the corresponding 3-vector form for
- $$v_1 = 1 + t^2 + t^4, \quad v_2 = -1 + t^4, \quad v_3 = -2 + t^2 + 4t^4.$$
- (c) Do the vectors v_1, v_2, v_3 above form a basis for PE_4 ?
(d) Given $v = t^4$, is $v \in \text{Span}\{v_1, v_2, v_3\}$?
6. For each question indicate whether you think the answer is TRUE or FALSE.
- (a) If \mathbf{A} is a 4×3 matrix, the linear system $\mathbf{Ax} = (0 \ 0 \ 0 \ 0)^T$ always has a solution.
(b) Let \mathbf{A} be an $n \times n$ matrix with n linearly independent columns. Then the reduced row echelon form of \mathbf{A} is the $n \times n$ identity matrix.

$$(c) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 4 & 5 \\ 5 & 6 \end{pmatrix}$$

(d) If \mathbf{A}, \mathbf{B} and \mathbf{C} are invertible square matrices, $((\mathbf{AB})^T \mathbf{C})^{-1} = \mathbf{C}^{-1}(\mathbf{A}^{-1})^T(\mathbf{B}^{-1})^T$.

(e) The initial value problem, $y' = 4y^{2/3}$, $y(0) = 0$ has a unique solution.

(f) The differential equation, $y' = e^{y-t^2-1}$ is separable.

(g) $y'' + e^t y = \sqrt{t}$ is a linear differential equation.

(h) $y(t) = e^{t^2}$ is a solution to $y'' - 2ty' - 2y = 0$.

(i) The general solution to $x'' + 2x' + 10x = 5 \cos(3t)$ has oscillations that become unbounded as $t \rightarrow \infty$.

(j) If $|\mathbf{AB}| = 3$ where \mathbf{A} and \mathbf{B} are both square matrices, then \mathbf{A} must be invertible.

7. Consider the following differential equation

$$ty'' - (t+2)y' + 2y = t^3$$

(a) Classify the differential equation in terms of the following:

- Order?
- Linear?
- Autonomous?
- Homogeneous?
- Constant coefficient?

(b) What is the homogeneous ODE? (Write it down.) Verify that $y_1(t) = e^t$ is a solution to the homogeneous equation.

(c) We will now construct another solution $y_2(t)$ to the homogeneous ODE. Let $y_2(t) = z(t)y_1(t)$, where $z(t)$ is a function that we will determine.

i. Substitute y_2 into the homogeneous equation from (b) to show that $z(t)$ must satisfy the ODE $tz'' + (t-2)z' = 0$.

ii. Use the change of variable $u = z'$ to obtain a first order ODE. Solve it for u .

iii. Now find $z(t)$. Conclude that $y_2(t) = -t^2 - 2t - 2$.

(d) Show that y_1 and y_2 span the solution space of the homogeneous equation. *Hint: You can solve this question and the rest of the problem even if you could not solve (c). You need the information provided in (c.iii).*

(e) What is the general solution to the homogeneous equation? What is the basis for the solution space of the homogeneous equation?

(f) Find a particular solution $y_p(t)$ to the nonhomogeneous equation.

(g) Combine your answers from (e) and (f) to find the general solution the nonhomogeneous equation.

8. Consider a forced simple harmonic oscillator with damping. Suppose the motion of the object is described by the ODE

$$x'' + 2x' + x = 25 \sin(2t)$$

- (a) We experimentally determine that it takes a force of 1 newton to displace the object 1 meter. Determine the mass of the object, the damping constant, and the spring constant.
 - (b) Is the corresponding homogeneous equation overdamped, critically damped, or underdamped? Justify your answer.
 - (c) Find the general solution of the nonhomogeneous equation.
 - (d) Let $x(0) = 0$ meters and $x'(0) = 1$ meter/sec. Find an expression for the displacement of the object for any time t .
 - (e) Describe the behavior of the system as time goes to infinity. Does the solution grow without bound?
 - (f) Consider the *homogeneous* equation for the remainder of this problem. Convert the homogeneous equation to a system of two first order equations and write the result in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$.
9. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & t & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

- (a) Show that $\lambda = 1$ is an eigenvalue of \mathbf{A} . Find the rest of the eigenvalues of \mathbf{A} . For each eigenvalue, what is its *algebraic* multiplicity?
 - (b) Let E_1 denote the eigenspace associated with the eigenvalue $\lambda = 1$.
 - i. For what values of t (if any) is E_1 0-dimensional? If possible, provide a basis for E_1 .
 - ii. For what values of t (if any) is E_1 1-dimensional? If possible, provide a basis for E_1 .
 - iii. For what values of t (if any) is E_1 2-dimensional? If possible, provide a basis for E_1 .
 - iv. For what values of t (if any) is E_1 3-dimensional? If possible, provide a basis for E_1 .
 - (c) Set $t = 1$. Find the *geometric* multiplicity of each eigenvalue you found in (a).
10. For each of the following, write down the form of the particular solution using the method of undetermined coefficients. If this is not possible, write NOT. DO NOT find the values of the coefficients; just write down the form.

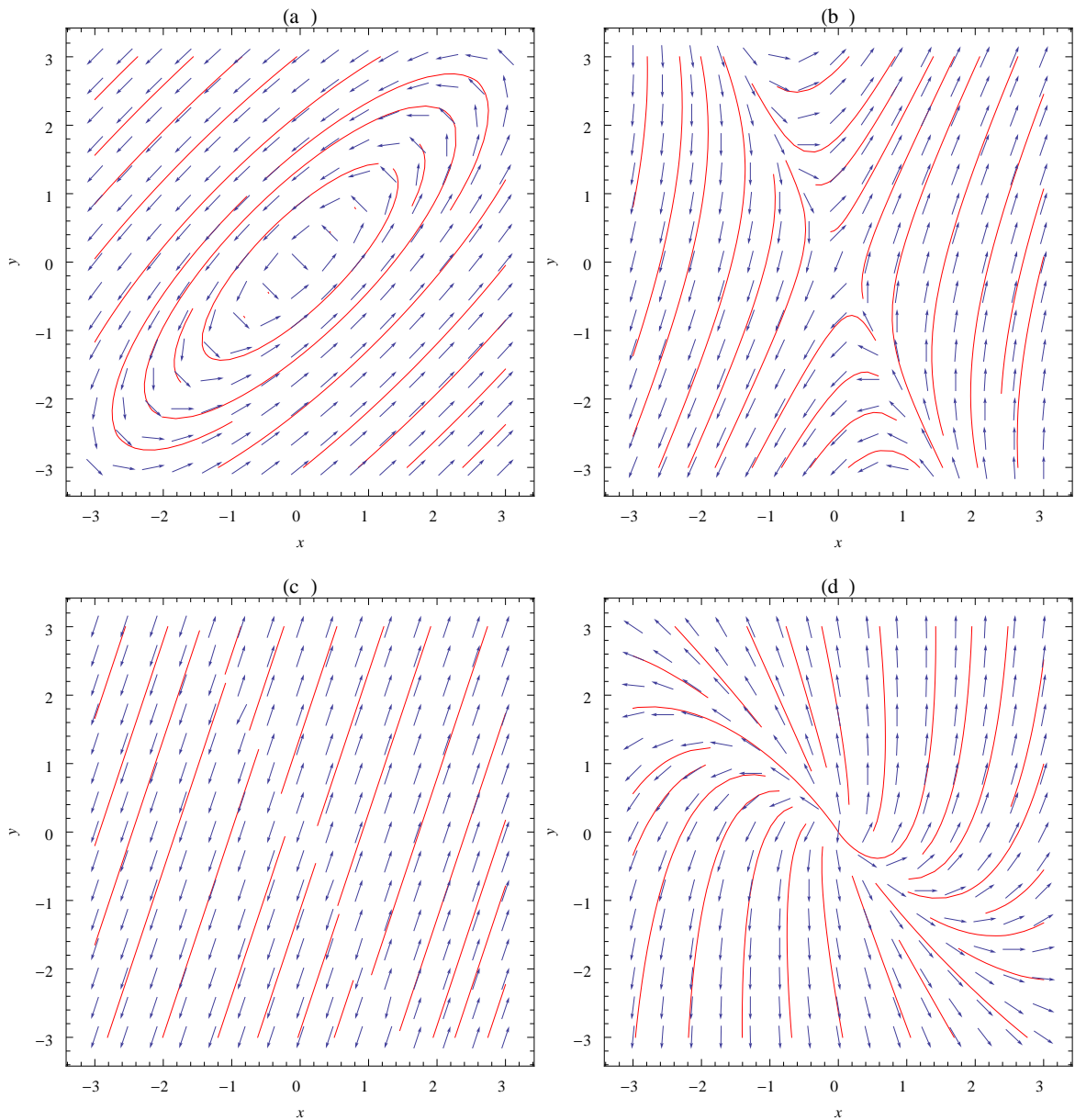
- (a) $y'' + 6y' + 9y = t(1 - t)$
- (b) $y'' + 6y' + 9y = e^{-3t}$
- (c) $y'' + 4y = \sin(t)$
- (d) $y'' = t(1 - t)$
- (e) $y'' + 2y' - 3y = t^2 - 3$
- (f) $y'' + 2y' - 3y = e^{t^2}$
- (g) $y'' + 2y' - 3y = \sin(t) + \cos(5t)$
- (h) $y'' + 2y' - 3y = t^3 e^{-3t}$
- (i) $y'' + 2y' - 3y = 4e^{-3t} \sin(t)$

11. Find the general solution to the system

$$\mathbf{x}' = \mathbf{A}\mathbf{x} = \begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix} \mathbf{x}$$

12. For each of the following equations, classify the equilibrium point at the origin and match the system with its phase portrait.

- i $\begin{cases} x' = 4x - 5y \\ y' = 5x - 4y \end{cases}$
- ii $\begin{cases} x' = x + y \\ y' = 4x + y \end{cases}$
- iii $\begin{cases} x' = 2x + y \\ y' = 6x + 3y \end{cases}$
- iv $\begin{cases} x' = 2x - y \\ y' = 4x + 6y \end{cases}$



13. Consider the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

- (a) Compute the determinant of \mathbf{A} .
- (b) Determine the eigenvalues and eigenvectors of \mathbf{A} .

(c) Solve the following IVP:

$$\begin{aligned}u' &= u - v \\v' &= -u + 2v - w \\w' &= -v + w \\u(0) &= 2, \quad v(0) = -1, \quad w(0) = 2\end{aligned}$$

(d) Give a basis for the solution space of the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

(e) Now find the eigenvalues and eigenvectors of

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

14. Short answer eigenstuff problems:

- (a) Let \mathbf{A} be an $n \times n$ matrix and \mathbf{B} be an invertible $n \times n$ matrix (*i.e.* there exists a matrix \mathbf{B}^{-1} such that $\mathbf{B}\mathbf{B}^{-1} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$). Show that \mathbf{A} and $\mathbf{B}^{-1}\mathbf{A}\mathbf{B}$ have the same eigenvalues. (You do not need to show this, but it is interesting to note that they do not (necessarily) have the same eigenvectors.)
- (b) Let \mathbf{A} be an $n \times n$ matrix and c be some constant. Show that \mathbf{A} and $\mathbf{A} - c\mathbf{I}$ have the same eigenvectors. What are the eigenvalues of $\mathbf{A} - c\mathbf{I}$?
- (c) Let \mathbf{A} be a 2×2 matrix with eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$. Find the eigenvalues of \mathbf{A}^5 .
- (d) If $\mathbf{v}_1 = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} b \\ d \end{pmatrix}$ are non-zero vectors such that $\mathbf{A}\mathbf{v}_1 = 2\mathbf{v}_1$ and $\mathbf{A}\mathbf{v}_2 = -\mathbf{v}_2$ for some 2×2 matrix \mathbf{A} , then what are the possible values of the rank of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$?
- (e) Let \mathbf{A} be a matrix, and let \mathbf{u}_1 and \mathbf{u}_2 be non-zero vectors such that $\mathbf{A}\mathbf{u}_1 = 2\mathbf{u}_1$ and $\mathbf{A}\mathbf{u}_2 = 2\mathbf{u}_2$. Which of the following statements are *necessarily* true:
- $\mathbf{u}_1 + \mathbf{u}_2$ is an eigenvector of \mathbf{A} with associated eigenvalue 4.
 - $3\mathbf{u}_1$ is an eigenvector of \mathbf{A} with associated eigenvalue 2.
 - $\det(\mathbf{A} - 2\mathbf{I}) = 0$.
- (f) Let \mathbf{A} be a 3×3 matrix with real entries. Which of the following statements could *possibly* be true:
- $\lambda_1 = 1 + i$ and $\lambda_2 = 3 - i$ are both eigenvalues of \mathbf{A} .
 - \mathbf{A} has only one eigenvalue, but three linearly independent eigenvectors.
 - All eigenvalues of \mathbf{A} have non-zero imaginary parts.

15. Consider the system

$$\begin{aligned}x' &= y \\y' &= -y + x - x^3\end{aligned}$$

- (a) Find all equilibria. Classify the stability of each equilibrium point.
- (b) Find and sketch the nullclines. Draw some typical solutions.
16. Solve the following.
- (a) (§2.3, problem 4) A radioactive material is known to decay at a rate proportional to the amount present. Over a 50-year period, an initial amount of 100 grams has decayed to only 75 grams. Find an expression for the amount of material t years after the initial measurement. Calculate the half-life of the material.
- (b) (§2.4, problem 6) A 600-gallon tank is filled with 300 gal of pure water. A spigot is opened and a salt solution containing 1 lb of salt per gallon of solution begins flowing into the tank at a rate of 3 gal/min. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave the tank at a rate of 1 gal/min. What will be the salt content in the tank at the precise moment that the volume of solution in the tank reaches the tank's capacity of 600 gal?
- (c) (§2.4, problem 20) Ryan always has a cup of coffee before his 8:00 AM recitation. Suppose the coffee is 200°F when poured from the coffee pot at 7:30 AM, and 15 minutes later it cools to 120°F in a room whose temperature is 70°F . Find an expression for the temperature of Ryan's coffee t minutes after 7:30 AM.