

APPM 2360: Section exam 3
7:15pm – 8:45pm, November 18, 2009.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

Problem 1: (20 points) Consider the second order differential equations

$$y'' + 2y' + 10y = 0$$

- (a) (10 points) Construct the general solution.
- (b) (4 points) Construct the particular solution with initial condition $y(0) = 1, y'(0) = 2$.
- (c) (6 points) Letting $x_1(t) = y(t)$ and $x_2(t) = y'(t)$, construct the linear system of two first order differential equations for $x_1(t)$ and $x_2(t)$, i.e., $\vec{x}' = \mathbf{A}\vec{x}$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. (Do not solve the linear system.)

Solution:

- (a) Characteristic equation: $r^2 + 2r + 10 = 0$. A pair of complex conjugate roots $r_{1,2} = -1 \pm 3i$. General solution: $y(t) = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t))$.
- (b) Note that $y'(t) = e^{-t}((-c_1 + 3c_2) \cos(3t) + (-c_2 - 3c_1) \sin(3t))$. Applying initial condition, a linear system is obtained:

$$c_1 = 1; -c_1 + 3c_2 = 2.$$

So $c_1 = 1$ and $c_2 = 1$. The particular solution is $y(t) = e^{-t}(\cos(3t) + \sin(3t))$.

(c)

$$\begin{aligned} x_1' &= x_2; \\ x_2' &= -10x_1 - 2x_2. \end{aligned}$$

Problem 2: (20 points)

- (a) (12 points) Use the *Method of Undetermined Coefficients* to write down the general form of the particular solution, y_p , for the differential equations given below but **do not solve**:

(i) $y'' - 3y' - 4y = -8e^{2t} \cos(3t)$ (ii) $y'' - 3y' - 4y = t^2 e^{-3t} \sin(t)$

(iii) $y'' - 3y' = 7 + 4y$ (iv) $y'' - 3y' - 4y = e^{-t}$

- (b) (8 points) Now use the *Method of Undetermined Coefficients* to find a particular solution of the equation,

$$y'' - 3y' - 4y = e^{-t}$$

Solution:

- (a) (i) $y_p = Ae^{2t} \cos(3t) + Be^{2t} \sin(3t)$
 - (ii) $y_p = (At^2 + Bt + C)e^{-3t} \sin(t) + (Dt^2 + Et + F)e^{-3t} \cos(t)$
 - (iii) $y_p = A$
 - (iv) $y_p = Ate^{-t}$, since $y = e^{-t}$ is a solution of the corresponding homogeneous equation. (Note $y = e^{4t}$ is also a solution of the corresponding homogeneous equation).
- (b) If $y_p = Ate^{-t}$, then $y_p' = Ae^{-t} - Ate^{-t}$, and $y_p'' = -2Ae^{-t} + Ate^{-t}$, now substituting y_p into the equation $y'' - 3y' - 4y = e^{-t}$ yields $A = -1/5$ and so $y_p = -\frac{1}{5}te^{-t}$.

PLEASE TURN OVER

Problem 3: (20 points) Consider the following nonhomogeneous Euler equation with initial conditions

$$t^2x'' + tx' - 4x = t^4 \quad x(1) = 0, \quad x'(1) = 0$$

- (a) (6 points) Find the general homogenous solution by assuming the form $x_h(t) = t^r$.
- (b) (9 points) Find the general solution to the nonhomogenous problem.
- (c) (5 points) Find the solution to the initial-value problem.

Solution:

- (a) The characteristic equation is given by $r(r-1) + r - 4 \equiv r^2 - 4 = 0$. Therefore $r_1 = 2, r_2 = -2$. The general homogenous solution is given by

$$x_h(t) = c_1t^2 + c_2t^{-2}$$

- (b) Solve the normalize problem

$$x'' + \frac{1}{t}x' - \frac{4}{t^2}x = t^2$$

by variation of parameters where $x_p(t) = v_1(t)t^2 + v_2(t)t^{-2}$. Therefore

$$\begin{bmatrix} t^2 & t^{-2} \\ 2t & -2t^{-3} \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t^2 \end{bmatrix}$$

The Wronskian for the homogenous solutions is $-4t^{-1}$. Using Cramer's rule:

$$v_1' = \frac{\begin{vmatrix} 0 & t^{-2} \\ t^2 & -2t^{-3} \end{vmatrix}}{-4t^{-1}} = \frac{1}{4}t, \quad v_2' = \frac{\begin{vmatrix} t^2 & 0 \\ 2t & t^2 \end{vmatrix}}{-4t^{-1}} = -\frac{1}{4}t^5$$

Therefore

$$v_1(t) = \frac{1}{8}t^2, \quad v_2(t) = -\frac{1}{24}t^6$$

Hence

$$x_p(t) = \frac{1}{8}t^2t^2 - \frac{1}{24}t^6t^{-2} = \frac{1}{12}t^4$$

The general solution is thus given by

$$x_g(t) = c_1t^2 + c_2t^{-2} + \frac{1}{12}t^4$$

- (c) Solve the initial value problem gives

$$\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} \\ -\frac{4}{12} \end{bmatrix}$$

Therefore

$$\begin{aligned} \left[\begin{array}{cc|c} 1 & 1 & -\frac{1}{12} \\ 2 & -2 & -\frac{4}{12} \end{array} \right] &\sim \left[\begin{array}{cc|c} 1 & 1 & -\frac{1}{12} \\ 1 & -1 & -\frac{1}{2} \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 1 & -\frac{1}{12} \\ 0 & 2 & \frac{1}{12} \end{array} \right] \sim \\ &\left[\begin{array}{cc|c} 1 & 1 & -\frac{1}{12} \\ 0 & 1 & \frac{1}{24} \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & 0 & -\frac{3}{24} \\ 0 & 1 & \frac{1}{24} \end{array} \right] \end{aligned}$$

Hence

$$x(t) = -\frac{3}{24}t^2 + \frac{1}{24}t^{-2} + \frac{1}{12}t^4 = -\frac{1}{8}t^2 + \frac{1}{24}t^{-2} + \frac{1}{12}t^4$$

Problem 4: (20 points) Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & -1 \end{bmatrix}$,

- (a) (10 points) Find the eigenvalues and eigenvectors of matrix \mathbf{A} .
 (b) (6 points) Find a general solution of the linear system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$.
 (c) (4 points) Give a basis for the solution space of the system $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$.

Solution:

(a) Since \mathbf{A} is upper triangular, $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$, and solving $\det(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \vec{\mathbf{0}}$ yields the corresponding eigenvectors $\vec{\mathbf{v}}_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, $\vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{v}}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(b) Note that each solution has the form $\vec{\mathbf{x}}_i = e^{\lambda_i t} \vec{\mathbf{v}}_i$, so from part (a) the general solution is

$$\vec{\mathbf{x}}(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ for any real numbers } c_1, c_2, c_3.$$

(c) The set $\left\{ e^{-t} \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the solution space.

Problem 5: (20 points) Consider the linear system of differential equations

$$\begin{cases} x' = x - 2y \\ y' = 2x + 5y \end{cases}.$$

- (a) (16 points) Construct the general solution to the above system.
 (b) (4 points) Construct the particular solution with initial condition $x(0) = 1, y(0) = -2$.

Solution:

(a)

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix},$$

$p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}| = \lambda^2 - 6\lambda + 9 = 0$ gives a double root $\lambda_1 = \lambda_2 = 3$. This eigenvalue has only one independent eigenvector: $\vec{\mathbf{v}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. To construct the general solution to the system, one needs to find a generalized eigenvector $\vec{\mathbf{u}}$ by solving $(\mathbf{A} - 3\mathbf{I})\vec{\mathbf{u}} = \vec{\mathbf{v}}$. One solution is $\vec{\mathbf{u}} = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}$. So the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \left(t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \right).$$

(b) Applying initial condition, one gets

$$\begin{aligned} c_1 &= 1, \\ -c_1 - c_2/2 &= -2. \end{aligned}$$

Solving the linear equations, one gets $c_1 = 1, c_2 = 2$. Plugging into the general solution, the particular solution is obtained

$$\begin{bmatrix} x \\ y \end{bmatrix} = e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2e^{3t} \left(t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/2 \end{bmatrix} \right).$$