APPM 2360: Exam 2

7:00pm – 8:30pm, November 17, 2010.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section (4) your instructor's name, and (5) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

Problem 1: (20 points) True/False. Please state whether the following are true, false Note that if they are ever untrue, they would be considered false.

(a) (4 points) If $\mathbf{Ax} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$, $\mathbf{A}$ has a zero eigenvalue.
(b) (4 points) Let $\mathbf{A}$ be a 2×2 matrix. If $det(\mathbf{A}) < \mathbf{0}$, $\mathbf{A}$ cannot have 2 positive real eigenvalues.
(c) (4 points) If a matrix has all positive eigenvalues, if all elements of $\mathbf{x}$ are greater than 0, all elements of $\mathbf{Ax}$ are greater than 0.
(d) If $\mathbf{A}$ has an eigenvalue of multiplicity 2, λ, $\mathbf{A^{-1}}$ has an eigenvalue λ^{-1}, of multiplicity 2.
(e) (4 points) If $\mathbf{A}$ has eigenvalues of 2 and 6 there is some vector for which $\mathbf{Ax} = \mathbf{4x}$.

Solution

(a) F
(b) T
(c) Either answer accepted.
(d) T
(e) F

Problem 2: (20 points) Suppose you have a spring with spring constant 72 N/m, and mass 2 kg.

(a) (4 points) Solve for the position and velocity of this physical system with no forcing or damping.
(b) (4 points) What is the natural frequency, ω_n, of this system? Solve for the position and velocity if the system is forced at $5\cos(\omega_n t)$. What is the name of the resulting behavior?
(c) (4 points) Suppose the mass is now placed on a table, leading to frictional damping, with b=24 kg/s. Solve for the resulting position and velocity when this system is not forced. Is the system, over, under or critically damped?

Solution

(a) $m\ddot{x} + kx = 0 \to 2\ddot{x} + 72x = 0$
$x(t) = c_1 \cos(\sqrt{k/m}t) + c_2 \sin(\sqrt{k/m}t) = c_1 \cos(6t) + c_2 \sin(6t)$
(b) Our natural frequency is ω_n=6. If we have a forcing term of $5\cos(6t)$, we expect a particular solution of the form $x_p(t) = c_1 t\cos(6t) + c_2 t \sin(6t)$ which gives

$$\begin{aligned}
\dot{x_p}(t) &= c_1 \cos(6t) - 6c_1 t \sin(6t) + c_2 \sin(6t) + 6c_2 \cos(6t) \\
\ddot{x_p}(t) &= -6c_1 \sin(6t) - 6c_1 t \sin(6t) - 36tc_1 \cos(6t) + 6c_2 \cos(6t) + 6c_2 \cos(6t) - 36tc_2 \sin(6t)
\end{aligned}$$

Plugging these into the differential equation and sorting by coefficients of terms, we get

$$t\cos(6t) : -72c_1 + 72c_1 = 0$$
$$t\sin(6t) : -72c_2 + 72c_2 = 0$$
$$\cos(6t) : 24c_2 = 5$$
$$\sin(6t) : 24c_1 = 0$$

The first two equations are useless, but the last two give $c_1 = \frac{5}{24}$ and $c_2 = 0$. This gives a particular solution of $\frac{5}{24}t\cos(6t)$ and a total solution of$x(t) = c_1\cos(6t) + c_2\sin(6t) + \frac{5}{24}t\cos(6t)$. This behavior is called pure resonance.

(c) With damping the equation becomes $2\ddot{x} + 24\dot{x} + 72x = 0$. The characteristic equation is thus $2r^2 + 24r + 72 = 0$ or $r^2 + 12r + 36 = 0$, which gives a double root of $r = -6$, giving the solution $x(t) = c_1e^{-6t} + c_2te^{-6t}$

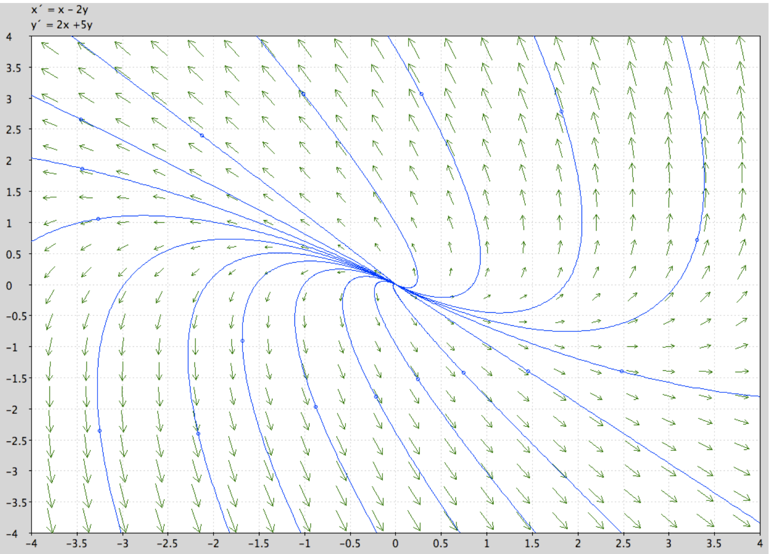
Problem 3: Consider the following system of differential equations

$$\begin{aligned} \dot{x_1} &= x_1 - 2x_2 \\ \dot{x_2} &= 2x_1 + 5x_2 \end{aligned}$$

(a) (8 points) Find two linearly independent solutions to this problem.
(b) (4 points) Sketch the phase portrait for this system. Identify and classify all equilibria.
(c) (4 points) Solve the system for the initial condition $x_1(0) = 2$ and $x_2(0) = 1$
(d) (4 points) What maximum value can the coefficient of x_2 in the second equation (currently 5) have and still give a stable equilibrium?

Solution

(a) The matrix is $\begin{pmatrix} 1 & -2 \\ 2 & 5 \end{pmatrix}$. We find the eigenvalues via $0 = \begin{vmatrix} 1-\lambda & -2 \\ 2 & 5-\lambda \end{vmatrix} = (5 - 6\lambda + \lambda^2) + 4 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$. Plugging this in to get the eigenvectors gives $\begin{pmatrix} -2 & -2 & | & 0 \\ 2 & 2 & | & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix}$ which gives the eigenvector $v = s[1, -1]^T$. To find a generalized eigenvector, we solve $\begin{pmatrix} -2 & -2 & | & 1 \\ 2 & 2 & | & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & | & -\frac{1}{2} \\ 0 & 0 & | & 0 \end{pmatrix}$, which gives $u = r[1, -1]^T - [0, \frac{1}{2}]^T$. The solution is thus $x(t) = c_1e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2e^{3t}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}t - \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}\right)$. Note that because the generalized eigenvector is not necessarily unique, it is possible to have slightly different answers for this question.



(b)

(c) Plugging in the initial condition, we have $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \left(- \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \right) = \begin{bmatrix} c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{c_2}{2} \end{bmatrix}$, which gives $c_1 = 2$ from the first equation, and $c_2 = -6$. Thus the final equation is $x(t) = 2e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 6e^{3t} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t - \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} \right)$

(d) The characteristic equation can be written as $\lambda^2 - Tr(A) + det(A) = 0$, which has roots $\frac{Tr(A) \pm \sqrt{Tr(A)^2 - 4det(A)}}{2} = \frac{1 + a_{22} - \sqrt{a_{22}^2 - 4(a_{22}+4)}}{2}$ The stability depends upon the real part of the eigenvalues, which must both be negative. Thus we must have a_{22} be smaller than -1 for the first term to be negative, which leaves the second term $\sqrt{1-20} = \sqrt{-19}$, which is imaginary. Therefore, it is necessary and sufficient for a_{22} to be less than -1.

Problem 4: (20 points) Consider the following second order ODE

$$y'' - \frac{1+t}{t} y' + \frac{1}{t} y = te^{2t}, \qquad 0 < t < \infty$$

(a) (8 points) Show that $y_1(t) = 1+t$ and $y_2(t) = e^t$ are two linearly independent homogeneous solutions to this problem.

(b) (12 points) Find the general solution.

Solution

(a) (8 points) Substitute into homogenous ODE.

$$(1+t)'' - \frac{1+t}{t}(1+t)' + \frac{1}{t}(1+t) = -\frac{1+t}{t} + \frac{1}{t}(1+t) = 0 \qquad (e^t)'' - \frac{1+t}{t}(e^t)' + \frac{1}{t}(e^t) = e^t(1 - \frac{1+t}{t} + \frac{1}{t}) = 0$$

Wronskian of $(1+t), e^t = te^t \neq 0$ therefore linearly independent.

(b) (12 points) By variation of parameters $y_p(t) = v_1(t)(1+t) + v_2(t)e^t$ where $v_1(t)$ and $v_2(t)$ satisfy

$$\begin{pmatrix} 1+t & e^t \\ 1 & e^t \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ te^{2t} \end{pmatrix}$$

By Cramers Rule

$$v_1' = -\frac{te^3 t}{te^t} = -e^{2t}, \qquad v_2' = \frac{(1+t)te^{2t}}{te^t} = (1+t)e^t$$

Therefore

$$v_1(t) = -\frac{1}{2}e^{2t}, \qquad v_2(t) = te^t$$

Therefore

$$y_p(t) = -\frac{1}{2}e^{2t} \cdot (1+t) + te^t \cdot e^t = \frac{1}{2}(t-1)e^{2t}$$

General solutoin

$$y_g(t) = c_1(1+t) + c_2 e^t + \frac{1}{2}(t-1)e^{2t}$$

Problem 5: (20 points) Consider the following linear system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$$

where $\mathbf{A}$ is a 2×2 matrix of real numbers.

(a) (6 points) Given the guess $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ show that λ and $\mathbf{v}$ satisfy the algebraic system

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

(b) (6 points) What are the conditions on $\mathbf{A}$ for complex eigenvalues and eigenvectors.

(c) (8 points) Assume that the solution to the algebraic system are complex conjugate pairs, i.e.

$$\begin{aligned} \lambda_1 &= \alpha + i\beta, \quad \mathbf{v}_1 = \mathbf{p} + i\mathbf{q} \\ \lambda_2 &= \alpha - i\beta, \quad \mathbf{v}_2 = \mathbf{p} - i\mathbf{q} \end{aligned}$$

where $\mathbf{p}$ and $\mathbf{q}$ are linearly independent vectors. Starting with $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, show/prove that the real-valued solutions are given by

$$\begin{aligned} \mathbf{u}(t) &= e^{\alpha t}\left(\cos(\beta t)\mathbf{p} - \sin(\beta t)\mathbf{q}\right) \\ \mathbf{w}(t) &= e^{\alpha t}\left(\sin(\beta t)\mathbf{p} + \cos(\beta t)\mathbf{q}\right) \end{aligned}$$

Solution

(a) If $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ then $\dot{\mathbf{x}}(t) = \lambda e^{\lambda t}\mathbf{v}$. Therefore $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ becomes $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ which is equivalent to $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$.

(b) $\mathrm{Tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) < 0$

(c)

$$\begin{aligned} \mathbf{x}(t) = e^{\lambda t}\mathbf{v} &= e^{(\alpha+i\beta)t}(\mathbf{p} + \mathbf{iq}) \\ &= e^{\alpha t}(\cos\beta t + i\sin\beta t)(\mathbf{p} + \mathbf{iq}) \\ &= e^{\alpha t}\left(\cos(\beta t)\mathbf{p} - \sin(\beta t)\mathbf{q}\right) + ie^{\alpha t}\left(\sin(\beta t)\mathbf{p} + \cos(\beta t)\mathbf{q}\right) \end{aligned}$$

Results follows upon taking the real and imaginary part.