APPM 2360: Final Exam

10:30am – 1:00pm, December 14, 2010.

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) recitation section (4) your instructor's name, and (5) a grading table. There are **EIGHT QUESTIONS** to be completed. Text books, class notes, and calculators are NOT permitted. Crib sheets are allowed.

Problem 1: (25 points)

(a) (10 points) Find the general solution to the differential equations

$$y' = \frac{t - e^{-t}}{y + e^{y}}$$

You may leave your solution in implicit form.

(b) It is often possible to solve a differential equation by making a change of dependent variable that converts it into a linear equation. One such equation is the Bernoulli equation which has the form

$$y' + p(t)y = q(t)y^n$$

For $n \neq 0, 1$ the substitution $v = y^{1-n}$ reduces Bernoulli's equation to the linear equation

$$v' + (1-n)p(t)v = (1-n)q(t)$$

In the following we consider the equation

$$t^2y' + 2ty - y^n = 0, \quad t > 0$$

Find the general solution to this equation

i) (5 points) when $n = 1$

ii) (10 points) when $n = 3$

Problem 2: (25 points) Consider the system

$$\begin{aligned} \dot{x} &= 20 - x^2 - y^2 \\ \dot{y} &= 8 - xy \end{aligned}$$

(a) (8 points) What are the equilibria for this system?

(b) (8 points) Classify each equilibrium for this system.

(c) (9 points) Sketch the phase plane, showing all equilibria, nullclines, and some possible solution curves. Please make your graph both clear and large.

Problem 3: (25 points) Consider the following problem. A 100 gallon tank starts out 75% full of fresh water. Water with a concentration of 1 lb salt/gal is pumped in at a rate of 1 gallon per minute, while water is pumped out of the tank at a rate of 3 gallons per minute.

(a) (10 points) What is the equation for the amount of salt in the tank at any time t?

(b) (5 points) How much salt is in the tank when it is half-full?

(c) (10 points) The input and output valves are redone so that the flow rate of input and output is now $\frac{t}{1+t^2}$. What is the equation for the amount of salt in the tank now?

Problem 4: (25 points) Consider the following matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -4 & 4 \\ 0 & 3 & 0 \\ -2 & 4 & -1 \end{pmatrix}$$

(a) (5 points) Does this matrix have an inverse?
(b) (10 points) Find the eigenvalues of $\mathbf{A}$.
(c) (10 points) Find the eigenvectors of $\mathbf{A}$.

Problem 5: (25 points)

(a) (10 points) Determine if the method of undetermined coefficients is applicable for each of the following differential equations.
(b) (15 points) If the answer is **YES**, find suitable forms of the particular solutions $y_p(t)$, but do not evaluate the constants.

i) $y'' - 2y' + y = 4te^{-t}$
ii) $y'' + 4y' - 4y = e^{2t}\left(1 + t^2\right)^{1/2}$
iii) $y'' - 8y' + 25y = e^{4t}\sin 3t$
iv) $y'' - 8y' + 25y = te^{2t}\cos t$
v) $y'' + 49y = 1 - \cos^2 7t$

Problem 6: (25 points) For what values of α and β does the system

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & \alpha & 2 \\ 0 & 2 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ \beta \\ 1 \end{pmatrix}$$

(a) (8 points) have 1 solution?
(b) (8 points) have 0 solutions?
(c) (9 points) have infinite solutions?

Problem 7: (25 points) Consider the following differential equation

$$-t(t-1)^2 = (t - t^2)y'' + (2t^2 - 1)y' + (2 - 4t)y$$

(a) (10 points) Verify that $y = t^2$ and $y = e^{2t}$ are linearly independent solutions to the associated homogeneous problem.
(b) (15 points) Find the general solution.

Problem 8: (25 points) Answer the following TRUE or FALSE. No justification is necessary.

i) According to Picard's Theorem the IVP $y' = y^{1/3}$, $y(0) = 4$ does not have a unique solution.

ii) If the timestep used in Euler's numerical intergration method for a first order initial value problem is reduced from h to $h/4$ then error in the approximation to solution $y(t)$ is reduced by a factor of 8.

iii) The set $V =$Span$\{1, x, \cos(e^x)\}$ is a vector space.

iv) The column space of the matrix

$$\begin{pmatrix} 1 & -3 & \frac{1}{2} & e \\ -2 & 6 & -1 & -2e \end{pmatrix}$$

is the line $y = -2x$ in R^2

v) Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of

$$\mathbf{A} = \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & e \end{pmatrix}$$

Then $\lambda_1\lambda_2\lambda_3 = 0$ and $\lambda_1 + \lambda_2 + \lambda_3 = (a + e)$.

vi) Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the eigenvalues of $\mathbf{A}$. Then $\mathbf{v}_1 = (1, 0, 0)^T, \mathbf{v}_2 = (0, 1, 0)^T, \mathbf{v}_3 = (0, 0, 1)^T$

vii The second order ODE $y" + \omega^2 y = G(y)$ where $G(y)$ is a nonlinear function of y is integrable.

viii) Let $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ be a 2×2 nonlinear system. Let (x_e, y_e) be an equilbrium solution to this system with a linearization that results in the eigenvalues $\lambda_1 = 2i, \lambda_2 = -2i$. It follows that the behaviour in the local neighborhood of the equilibrium point is a center.