APPM 2360: Exam 3

7:00pm – 8:30pm, April 14, 2010.

ON THE FRONT OF YOUR BLUEBOOK write the following: (1) name, (2) Test 3/instructors name, (3) recitation section, and (4) a grading table. Text books, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

Problem 1: (20 points)

(a) (6 points) Suppose A is a 3×3 matrix with eigenvalue -2 and corresponding eigenvector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Find one eigenvalue and eigenvector for the matrix A^7.

(b) (7 points) Suppose A is a 2×2 matrix with eigenvalues 1 and -1 and corresponding eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Let

$$A\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Find $\vec{x}$.

(c) (7 points) Assume that for any $n \times n$ matrix A with real entries, if $A = -A^T$, then the eigenvalues of A are purely imaginary, or have no real part.

(i) (4 points) Suppose for a real 2×2 matrix A, $A = -A^T$. What is the sum of the eigenvalues of A? Please explain how you found the sum.

(ii) (3 points) Suppose for a real 3×3 matrix A, $A = -A^T$. You should now immediately know one of the eigenvalues of A. What is it and why?

Solution

(a) $Av = \lambda v$ implies $A^7 v = \lambda^7 v$. Thus, an e-val of A^7 is -128, and an eigenvector is the one given.

(b) Since the e-vals are different, the eigenvectors are l.i. and therefore form a basis for the space. Let

$$\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Then

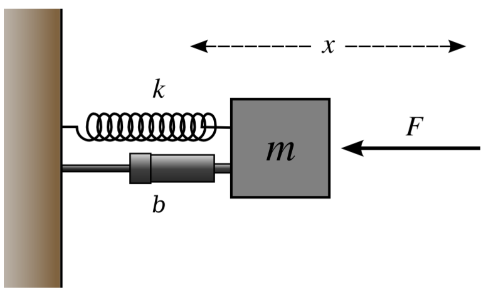
$$A\vec{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

and thus we need to solve the system

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This gives $c_1 = c_2 = \frac{1}{2}$, and thus $\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(c) For both parts, since A has only real entries, eigenvalues must come in conjugate pairs. Thus for the first part, our e-vals are $\pm\alpha i$, and thus the sum is zero. In the second case, since 3 is odd, 0 must be an eigenvalue since it is its own conjugate, else we would have four evals.



Problem 2: (20 points) Given a mass-spring system with mass $m = 1/4$, damping coefficient $b = 1$, spring constant $k = 21$, horizontal displacement from rest x, and external forcing $F(t) = -\sin(2t) + 2\cos(2t)$ (see the picture), answer the following questions.

a) (3 points) Is this system over-damped, critically damped, or under-damped?

b) (5 points) For the initial condition $x_h(0) = 2$ and $\dot{x}_h(0) = 0$, draw a plot of the shape of the homogeneous solution $x_h(t)$ as a function of time and (in words) describe the governing features.

b) (12 points) Find the particular solution $x_p(t)$ using the Method of Undetermined Coefficients.

Solution:

a) The system is under-damped because the discriminant is -20, which is less than zero.

b) The homogenous solution is

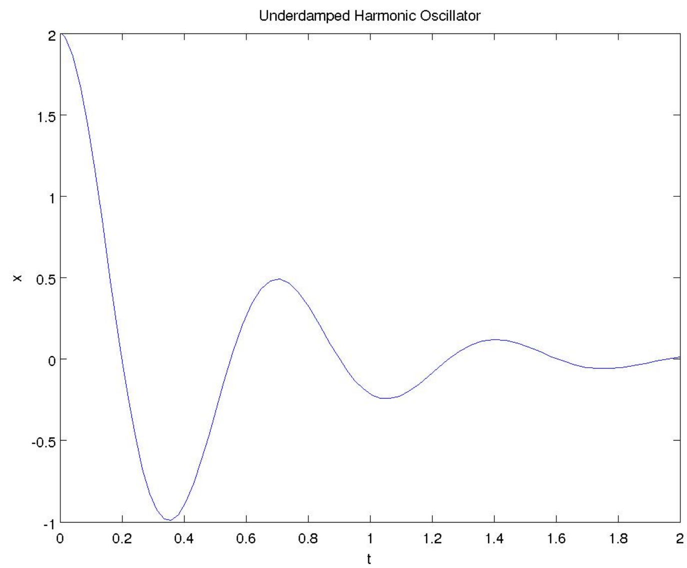
$$x_h(t) = e^{-2t}(c_1 \cos(4\sqrt{5}t) + c_2 \sin(4\sqrt{5}t))$$

For the given initial condition, we have $c_1 = 2$ and $c_2 = 1/\sqrt{5}$.

Here is a plot of the homogeneous solution $x_h(t)$ for initial conditions $x(0) = 2$ and $\dot{x}(0) = 0$. For full credit, the plot and the caption must describe oscillations with an exponentially decaying amplitude.

c) Using the Method of Undetermined Coefficients, we start with the expansion

$$x_p = A\cos(2t) + B\sin(2t)$$



At the end of the day, this should lead to the linear system

$$\begin{pmatrix} 20 & 2 \\ -2 & 20 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Solving this gives

$$A = 42/404 = 21/202, \ B = -16/404 = -8/202 = -4/101$$

Problem 3:

Let

$$t^2 \frac{d^2y}{dt^2} + 5t\frac{dy}{dt} + 3y = t^{5/2}, \ t > 0.$$

(a) (10 points) Using the guess $y = t^\lambda$, find two homogenous solutions to the above ODE. You should get two values for λ which are -1 and -3. If you cannot show this, you will **LOSE** 6 points. Now, show these two solutions are linearly independent. Note, your statement of linear independence should be a complete sentence.

(b) (10 points) Find the particular solution to the forced problem using Variation of Parameters. Once you find the integrals, you do **NOT** need to simplify your solution any further.

Solution:

(a) We see from our guess that

$$\lambda(\lambda - 1) + 5\lambda + 3 = 0.$$

Thus $\lambda = -1, -3$, and we get two homogenous solutions $y_1(t) = t^{-1}$ and $y_2(t) = t^{-3}$. The Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = -2t^{-5}.$$

Since this is never zero, we know that y_1 and y_2 are linearly independent.

(b) Dividing the differential equation by t^2 gives a right-hand side of $t^{1/2}$. Thus we need to find the integrals

$$\int \frac{-y_2 f}{W} dt = \frac{1}{2} \int t^2 t^{1/2} dt = \frac{1}{7} t^{7/2},$$

and

$$\int \frac{y_1 f}{W} dt = \frac{-1}{2} \int t^4 t^{1/2} dt = -\frac{1}{11} t^{11/2},$$

and so our particular solution is

$$y_p(t) = \frac{1}{7t} t^{7/2} - \frac{1}{11t^3} t^{11/2},$$

or

$$y_p = \frac{1}{7} t^{5/2} - \frac{1}{11} t^{5/2}.$$

Problem 4: (20 points) Given the matrix $\mathbf{A} = \begin{bmatrix} 1 & 0 & -2 \\ -4 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$,

(a) (8 points) Show that 1 is an eigenvalue of the matrix $\mathbf{A}$. Then find the other eigenvalues.

(b) (12 points) For the eigenvalue -1, find the corresponding eigenvectors and determine the dimension and basis of the corresponding eigenspace.

Solution:

(a)

$$p(\lambda) = \begin{vmatrix} 1-\lambda & 0 & -2 \\ -4 & -1-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)^2 = 0$$

yields that $\lambda_1 = 1, \lambda_2 = \lambda_3 = -1$.

(b) Solving the system $(\mathbf{A} - \lambda\mathbf{I})\vec{\mathbf{v}} = \vec{\mathbf{0}}$ gives the corresponding eigenvector(s).

For $\lambda_2 = \lambda_3 = -1$,

$$\left[\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution: $\vec{\mathbf{v}} = r \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, r, s \in \mathbb{R}$.

Two independent eigenvectors: $\vec{\mathbf{v}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{\mathbf{v}}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

$\mathbb{E}_{-1} = \text{Span}\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$.

$\dim(\mathbb{E}_{-1}) = 2$; a basis $\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \}$.

Problem 5: Let

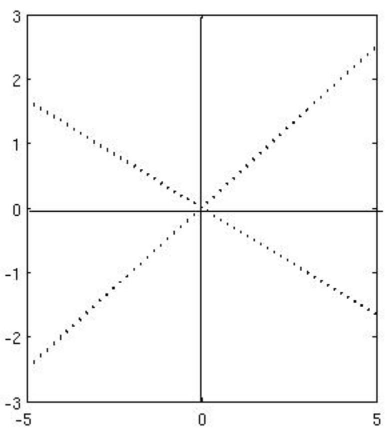
$$\frac{d\vec{x}}{dt} = \begin{pmatrix} -1 & 12 \\ 2 & 1 \end{pmatrix} \vec{x}$$

(a) (10 points) Find the characteristic equation to the above matrix, and then find its roots (*i.e.* eigenvalues).

(b) (10 points) Classify the type of equilibrium point at the origin (*i.e.* saddle or node). Given that the eigenvectors can be written as

$$\begin{pmatrix} -3 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

write the general solution to the system above. Now, in the figure below, the eigenspaces are given by the dotted lines. Copy this figure into your blue book, and then indicate with arrows in which direction the flow moves on each eigenspace. Between each of the dotted lines, draw **one** curve with arrows on it denoting the direction of the flow. You do not need to plot the function *per se*, but instead give a general idea of the behavoir of a solution in the given region.



Solution:

The characteristic equation for the matrix is $\lambda^2 - 25 = 0$, and thus the eigenvalues are $\lambda = \pm 5$. Thus the equilibrium is a saddle point. Applying the given matrix to the given eigenvectors shows that the general solution to the differential equation is

$$\vec{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + c_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} e^{-5t}.$$

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