
INSTRUCTIONS: ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) "Final"/instructors name, (3) recitation section, (4) and a grading table. Textbooks, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

This final contains 8 wonderful problems. Good luck!

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1. [26] Consider the linear system

$$\begin{cases} x_1 + 3x_2 + x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 - 14x_3 + 2x_4 &= 6 \\ x_1 + 11x_2 + 17x_3 + x_4 &= 3 \end{cases}$$

This system can be written in matrix-vector form as $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$:

(a) [5] Write down the matrix A, and the vector $\vec{\mathbf{b}}$ in this case.

(b) [9] Find the general solution to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

(c) [12] Find the general solution to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

2. [25] Eigenstuff

(a) [5] Given

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

find the eigenvalues and eigenvectors of A.

(b) [5] Given

$$B = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

find the eigenvalues and eigenvectors of B.

(c) [8] Find the eigenvalues and eigenvectors of AB. Show every eigenvalue of AB is a product of the eigenvalues of A and B. *Hint: What do A and B share in common?*

(d) [7] Find the eigenvalues and eigenvectors of AB^2A^3. *Hint: Again, what do A and B share in common?*

3. [25] Give a brief answer to each question. Show all work.

   (a) [5] Let $F = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 2 \\ -2 & -3 & -4 \end{bmatrix}$, and let $M = \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{bmatrix}$, verify that $\boxed{M = (F^{-1})^T}$.
   Hint: Rearrange this equation using matrix operation properties and then show the equality is true.

   (b) [5] Given $B = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$, find a 2×2 matrix K such that $D = KB$.

   (c) [15] Let F and D be 3×3 matrices with $\det(F) = 10$, $\det(D) = 6$ and $\det(F + D) = 90$ find the values of

      i. $\det(F + F + F)$
      ii. $\det(F^2 D + F D^2)$
      iii. $\det\left((FD)^{-1} F^T F\right)$

4. [26] Suppose you have a 1 kilogram mass, a spring with spring constant 10 Newtons/meter and frictional coefficient $b = 2$ Newton seconds/meter.

   (a) [5] Using $x(t)$ to measure displacement from equilibrium for the homogeneous mass-spring system associated with the above constants and the initial condition $x(0) = 0$ and $\dot{x}(0) = 3$.

      i. Write down the solution $x(t)$.
      ii. What kind of damped system is this?

   (b) [5] For a mass-spring system with damping (and no forcing), the displacement $x(t)$ decays to zero as $t \to \infty$. If we want to solve the differential equation on a computer, however, there are situations when computers set numbers less than 10^{-16} to zero. What is the critical value of t after which your solution to part (a) is less than 10^{-16} and hence vulnerable to being reported as zero by a computer? *Note: You don't have to simplify your answer. Hint: recall that* $|\sin(\alpha t)| \leq 1$.

   (c) [5] Now suppose, starting from rest, you have a forcing $F(t) = F_0 \cos(\omega_f t)$ with $\omega_f = \sqrt{11} - 1$ and F_0 chosen so that we get the particular solution

$$x_p(t) = \cos(\omega_f t - \delta)\,,$$

   with

$$\tan(\delta) = \frac{b\omega_f}{m(\omega_0^2 - \omega_f^2)}\,,$$

   where $\omega_0 = \sqrt{k/m}$ is the natural frequency of oscillation. Find δ.

   (d) [6] Introduce a new forcing function $F(t) = \tilde{F}_0 \cos(\tilde{\omega}_f t)$ with $\tilde{F}_0$ chosen such that the particular solution for this forcing function is

$$\tilde{x}_p(t) = \cos(\tilde{\omega}_f t - \tilde{\delta})\,.$$

   How should one choose a positive $\tilde{\omega}_f$ so that the new system $\tilde{x}_p(t)$ is 90 degrees out of phase with the first system $x_p(t)$, i.e., $|\delta - \tilde{\delta}| = \pi/2$?

(e) [5] Consider a forcing function which is the sum of the forcings from parts (c) and (d), i.e., $F(t) = F_0 \cos(\omega_f t) + \tilde{F}_0 \cos(\tilde{\omega}_f t)$. Find the steady state equilibrium solution, i.e., find

$$\lim_{t \to \infty} x(t) \,,$$

where $x(t)$ is the solution to the damped, forced mass-spring system.

5. [24] True or False. State whether the following statements are (always) "TRUE" or "FALSE" (meaning not always true). You MUST write the full word TRUE or FALSE — T/F will NOT be graded. Each correct answer earns 3 points. For this question only you do NOT need to show your working or reasoning.

(a) Suppose $\vec{\mathbf{v}}_1,\ \vec{\mathbf{v}}_2,\ \vec{\mathbf{v}}_3$ are three non-zero $\mathbb{R}^7$ vectors and they are linearly independent. Then $\{\vec{\mathbf{v}}_1,\ \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2,\ \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_3\}$ must be linearly independent.

(b) All solutions to the differential equation $y'' - \cos t^2 y' + y = 5t$ form a vector space (usual addition and scalar multiplication are assumed).

(c) For linear system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A}$ is a 2×2 matrix, if $|\mathbf{A}| = -3$, then equilibrium $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is always a saddle.

(d) The origin of the linear system $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -8 \\ 5 & 1 \end{bmatrix} \vec{\mathbf{x}}$ is a spiral. Then nearby trajectories rotate with a counterclockwise direction in the phase plane.

(e) $y_1(t) = \frac{5}{\sin(t)+\cos(2t)}$ is the solution to the IVP $y' + \sin(t)y = \cos(2t),\ y(0) = 2$.

(f) Picard's theorem guarantees that the IVP $y' - 2y^{2/3} = 0,\ y(0) = 0$ has at least one solution.

(g) The determinant of the matrix $\begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 3 & -3 & 0 \\ 1 & 3 & 0 & 3 \\ 2 & 3 & 6 & 0 \end{bmatrix}$ is $\frac{1}{3}$.

(h) The dimension of $\text{Span}\{e^t, e^{2t}, e^t + 3e^{2t}, e^t - 2e^{2t}\}$ is 4.

6. [25] Consider the differential equation

$$y'' - 5y' + 4y = 3e^t.$$

(a) [6] Find the general solution to the corresponding homogeneous equation.

(b) [9] Construct a particular solution to the non-homogeneous equation.

(c) [4] Find the solution with initial condition $y(0) = 2, y'(0) = 4$.

(d) [6] Use the Method of Undetermined Coefficients to write down the general form of the particular solution, y_p, for the differential equations given below but do **NOT** solve.

i. $y'' - 5y' + 4y = 3\cos(t)e^{4t}$

ii. $y'' - 5y' + 4y = t^2 e^{4t}$

iii. $y'' - 5y' + 4y = t^3$

7. [25] Short answer

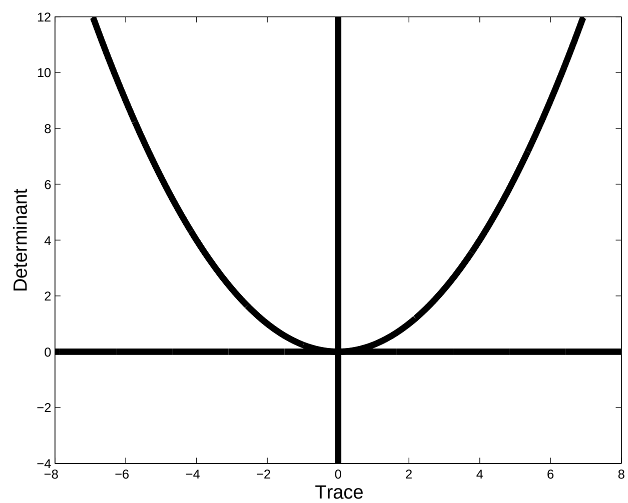
(a) [9] Which of the following systems have unstable equilibrium points at zero?

i. $y' = y$

ii. $y' = -2y + 2y^2$

iii. $y' = y(y-1)(y-2)$

(b) [16] Below is the trace/determinant plane for $\bar{x}' = A\bar{x}$ where A is any 2×2 matrix. For your convenience, the figure also contains curves that define the boundaries of behavior for different solutions.



Let A be the matrix

$$A(a) = \begin{bmatrix} a & 1 \\ a-2 & 1 \end{bmatrix}.$$

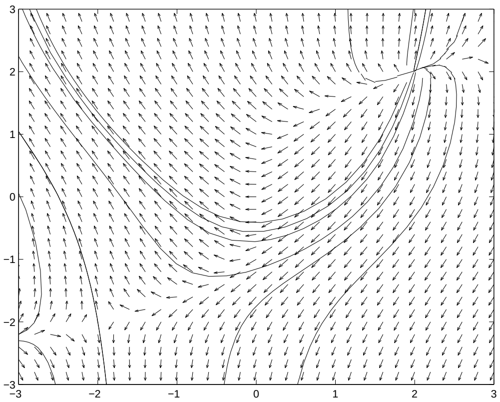
where two of the elements of A depend on the value of a parameter a. If a can be any real value, describe the different classifications that the steady state node $\bar{x}_* = (0,0)$ can have.

8. [24] Match each system with its phase portrait below. (You don't need to show your work.)

$$1.\begin{cases} x' &= x+y \\ y' &= -2x-y \end{cases} \quad 2.\begin{cases} x' &= -x+5y \\ y' &= x-3y \end{cases} \quad 3.\begin{cases} x' &= y \\ y' &= -x\sin(x) \end{cases}$$

$$4.\begin{cases} x' &= x\sin(y) \\ y' &= y\cos(x) \end{cases} \quad 5.\begin{cases} x' &= xy-4 \\ y' &= y^3-4x \end{cases} \quad 6.\begin{cases} x' &= y-x \\ y' &= x^2+y^2-8 \end{cases}$$

(A)



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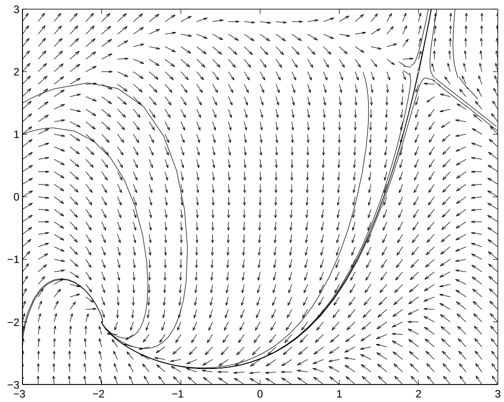

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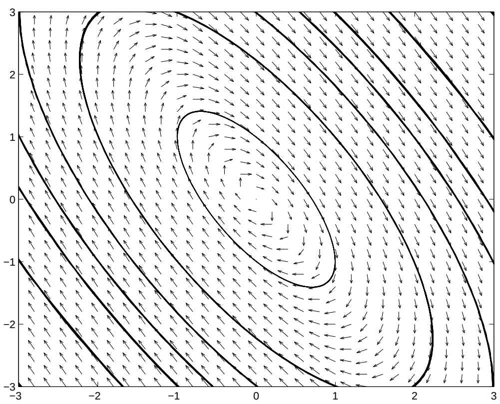

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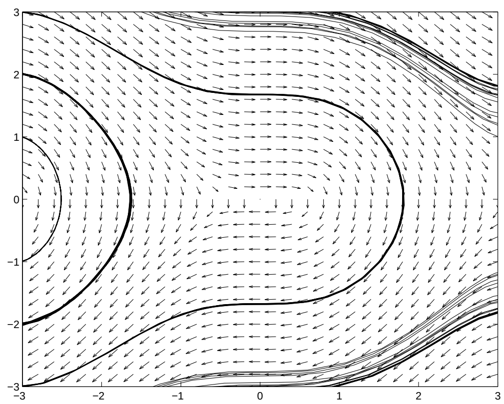

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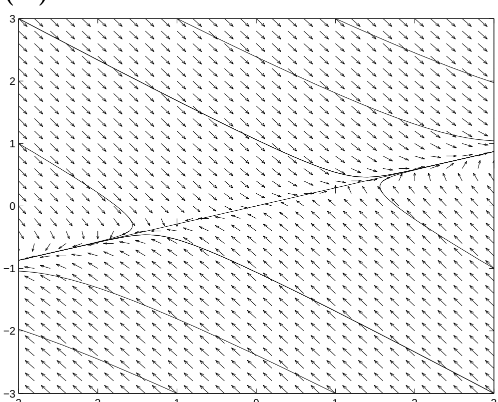

(F)