APPM 2360 **FINAL EXAM** May 3, 2010

INSTRUCTIONS: ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) "Final"/instructors name, (3) recitation section, (4) and a grading table. Textbooks, class notes, and calculators are NOT permitted. A one-page crib sheet is allowed.

This final contains 8 wonderful problems. Good luck!

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1. [26] Consider the linear system

$$\begin{cases} x_1 + 3x_2 + x_3 + x_4 &= 3 \\ 2x_1 - 2x_2 - 14x_3 + 2x_4 &= 6 \\ x_1 + 11x_2 + 17x_3 + x_4 &= 3 \end{cases}$$

This system can be written in matrix-vector form as $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$:

(a) [5] Write down the matrix A, and the vector $\vec{\mathbf{b}}$ in this case.

(b) [9] Find the general solution to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$.

(c) [12] Find the general solution to the equation $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

Solution:

(a) We have

$$A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & -2 & -14 & 2 \\ 1 & 11 & 17 & 1 \end{bmatrix}, \qquad \vec{\mathbf{b}} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix}.$$

(b) The augmented matrix is

$$[\,A\,|\,\vec{\mathbf{b}}\,] = \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 2 & -2 & -14 & 2 & 0 \\ 1 & 11 & 17 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & -8 & -16 & 0 & 0 \\ 0 & 8 & 16 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 0 \\ 0 & -8 & -16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So x_3 and x_4 are free variables and the general solution is

$$\vec{\mathbf{x}} = \begin{pmatrix} 5 \\ -2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} t, \quad \text{for any } s, t \in \mathbb{R}.$$

(c) The augmented matrix is

$$[\,A\,|\,\vec{\mathbf{b}}\,] = \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 2 & -2 & -14 & 2 & 6 \\ 1 & 11 & 17 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & -8 & -16 & 0 & 0 \\ 0 & 8 & 16 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 3 \\ 0 & -8 & -16 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So the general solution is

$$\vec{\mathbf{x}} = \begin{pmatrix} 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ -2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} t, \quad \text{for any } s, t \in \mathbb{R}.$$

2. [25] Eigenstuff

   (a) [5] Given

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

   find the eigenvalues and eigenvectors of A.

   (b) [5] Given

$$B = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

   find the eigenvalues and eigenvectors of B.

   (c) [8] Find the eigenvalues and eigenvectors of AB. Show every eigenvalue of AB is a product of the eigenvalues of A and B. *Hint: What do A and B share in common?*

   (d) [7] Find the eigenvalues and eigenvectors of AB^2A^3. *Hint: Again, what do A and B share in common?*

Solutions:

(a) A is upper triangular, so the eigenvalues are $\lambda = 1, 2$. We see for $\lambda = 1$ the e-vector must be $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. For $\lambda = 2$, we get the e-vectors $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

(b) B is upper triangular, so the eigenvalues are $\lambda = 1, 3$. The eigenvectors of B are the same as for A.

(c) A and B share eigenvectors, so if $Av = \lambda v$ and $Bv = \gamma v$ then $ABv = \lambda\gamma v$. So evals of AB are 1 and 6. The evectors again are unchanged.

(d) E-vectors remain unchanged. E-vals must be 1 and $144 = (2)(3)^2(2)^3$.

3. [25] Give a brief answer to each question. Show all work.

(a) [5] Let $F = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 2 \\ -2 & -3 & -4 \end{bmatrix}$, and let $M = \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{bmatrix}$, verify that $\boxed{M = (F^{-1})^T}$.
Hint: Rearrange this equation using matrix operation properties and then show the equality is true.

(b) [5] Given $B = \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}$, find a 2×2 matrix K such that $D = KB$.

(c) [15] Let F and D be 3×3 matrices with $\det(F) = 10$, $\det(D) = 6$ and $\det(F + D) = 90$ find the values of

i. $\det(F + F + F)$
ii. $\det(F^2 D + F D^2)$
iii. $\det\left((FD)^{-1} F^T F\right)$

Solution:

(a) $F^T M = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{bmatrix} \begin{bmatrix} 14 & -8 & -1 \\ -17 & 10 & 1 \\ -19 & 11 & 1 \end{bmatrix} = I = MF^T$, so $M = (F^T)^{-1} = (F^{-1})^T$.

(b) $K = DB^{-1} = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} 14 & -32 \\ -6 & 16 \end{bmatrix}$

(c)

i. $\det(3F) = \det(3I \cdot F) = \det(3I)\det(F) = 3^3 \cdot 10 = 270$
ii. $\det(F^2 D + F D^2) = \det\Big(F \cdot (F + D) \cdot D\Big) = \det(F)\det(F + D)\det(D) = 10 \cdot 90 \cdot 6 = 5400$
iii. $\det((FD)^{-1} F^T F) = \det(F^T F)/\det(FD) = \det(F)^2/\det(F)\det(D) = 10/6 = 5/3$

4. [26] Suppose you have a 1 kilogram mass, a spring with spring constant 10 Newtons/meter and frictional coefficient $b = 2$ Newton seconds/meter.

(a) [5] Using $x(t)$ to measure displacement from equilibrium for the homogeneous mass-spring system associated with the above constants and the initial condition $x(0) = 0$ and $\dot{x}(0) = 3$.

i. Write down the solution $x(t)$.
ii. What kind of damped system is this?

(b) [5] For a mass-spring system with damping (and no forcing), the displacement $x(t)$ decays to zero as $t \to \infty$. If we want to solve the differential equation on a computer, however, there are situations when computers set numbers less than 10^{-16} to zero. What is the critical value of t after which your solution to part (a) is less than 10^{-16} and hence vulnerable to being reported as zero by a computer? *Note: You don't have to simplify your answer. Hint: recall that* $|\sin(\alpha t)| \leq 1$.

(c) [5] Now suppose, starting from rest, you have a forcing $F(t) = F_0 \cos(\omega_f t)$ with $\omega_f = \sqrt{11} - 1$ and F_0 chosen so that we get the particular solution

$$x_p(t) = \cos(\omega_f t - \delta)\,,$$

with

$$\tan(\delta) = \frac{b\omega_f}{m(\omega_0^2 - \omega_f^2)},$$

where $\omega_0 = \sqrt{k/m}$ is the natural frequency of oscillation. Find δ.

(d) [6] Introduce a new forcing function $F(t) = \tilde{F}_0 \cos(\tilde{\omega}_f t)$ with $\tilde{F}_0$ chosen such that the particular solution for this forcing function is

$$\tilde{x}_p(t) = \cos(\tilde{\omega}_f t - \tilde{\delta}).$$

How should one choose a positive $\tilde{\omega}_f$ so that the new system $\tilde{x}_p(t)$ is 90 degrees out of phase with the first system $x_p(t)$, i.e., $|\delta - \tilde{\delta}| = \pi/2$?

(e) [5] Consider a forcing function which is the sum of the forcings from parts (*c*) and (*d*), i.e., $F(t) = F_0 \cos(\omega_f t) + \tilde{F}_0 \cos(\tilde{\omega}_f t)$. Find the steady state equilibrium solution, i.e., find

$$\lim_{t\to\infty} x(t),$$

where $x(t)$ is the solution to the damped, forced mass-spring system.

Solutions:

(a) We get the differential equation

$$\ddot{x} + 2\dot{x} + 10x = 0$$

This gives the general solution

$$x(t) = e^{-t}(c_1 \cos(3t) + c_2 \sin(3t))$$

and the system is underdamped. With the initial conditions, we get $x(t) = e^{-t}\sin(3t)$.

(b) We want $e^{-t} \leq 10^{-16}$ or $t \geq 16\ln(10)$.

(c) Plugging in $\omega_f = \sqrt{11} - 1$, $\omega_0^2 = 10$, gives $\tan(\delta) = 1$ or $\delta = \pi/4$.

(d) We want $\delta = 3\pi/4$, or $\tan(\delta) = -1$. Solving for ω_f, we get $\omega_f = \sqrt{11} + 1$.

(e) The answer must be: $\cos((\sqrt{11} - 1)t - \pi/4) + \cos((\sqrt{11} + 1)t - 3\pi/4)$.

5. [24] True or False. State whether the following statements are (always) "TRUE" or "FALSE" (meaning not always true). You MUST write the full word TRUE or FALSE — T/F will NOT be graded. Each correct answer earns 3 points. For this question only you do NOT need to show your working or reasoning.

(a) Suppose $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ are three non-zero $\mathbb{R}^7$ vectors and they are linearly independent. Then $\{\vec{\mathbf{v}}_1,\ \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2,\ \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_3\}$ must be linearly independent.

(b) All solutions to the differential equation $y'' - \cos t^2 y' + y = 5t$ form a vector space (usual addition and scalar multiplication are assumed).

(c) For linear system of differential equations $\vec{\mathbf{x}}' = \mathbf{A}\vec{\mathbf{x}}$ where $\mathbf{A}$ is a 2×2 matrix, if $|\mathbf{A}| = -3$, then equilibrium $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is always a saddle.

(d) The origin of the linear system $\vec{\mathbf{x}}' = \begin{bmatrix} 1 & -8 \\ 5 & 1 \end{bmatrix} \vec{\mathbf{x}}$ is a spiral. Then nearby trajectories rotate with a counterclockwise direction in the phase plane.

(e) $y_1(t) = \frac{5}{\sin(t)+\cos(2t)}$ is the solution to the IVP $y' + \sin(t)y = \cos(2t),\ y(0) = 2$.

(f) Picard's theorem guarantees that the IVP $y' - 2y^{2/3} = 0,\ y(0) = 0$ has at least one solution.

(g) The determinant of the matrix $\begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 3 & -3 & 0 \\ 1 & 3 & 0 & 3 \\ 2 & 3 & 6 & 0 \end{bmatrix}$ is $\frac{1}{3}$.

(h) The dimension of $\text{Span}\{e^t, e^{2t}, e^t + 3e^{2t}, e^t - 2e^{2t}\}$ is 4.

Solution:

(a) True

(b) False

(c) True

(d) True

(e) False

(f) True

(g) False

(h) False

Comments:

(a) $C_1\vec{\mathbf{v}}_1 + C_2(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2) + C_3(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_3) = \vec{\mathbf{0}}$ implies that $C_1 + C_2 + C_3 = 0,\ C_2 + C_3 = 0,\ C_3 = 0$. This linear system only has the zero solution so $\{\vec{\mathbf{v}}_1,\ \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2,\ \vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_3\}$ must be linearly independent.

(b) Apparently, $y_1 + y_2$ is not a solution if y_1 and y_2 are two solutions.

(c) The two eigenvalues must be one positive and one negative since $\lambda_1\lambda_2 = |\mathbf{A}| = -3$.

(d) The direction at the point $(1,0)$ is $(1,5)$. The trajectory moves up to the right when passing through the point $(1,0)$. So nearby trajectories have counterclockwise motions.

(e) $y_1(0) = 5 \neq 2$, so it is not a solution.

(f) $f(t,y) = 2y^{2/3}$ is continuous in the neighborhood of $(0,0)$. So the first part of Picard's theorem applies and it gives the existence of the solutions.

(g) No division operation is used when computing determinants of matrices. So no fractions can be produced if all entries are integers.

(h) The dimension should be 2, since the last two functions are the linear combinations of the fist two functions.

6. [25] Consider the differential equation

$$y'' - 5y' + 4y = 3e^t.$$

(a) [6] Find the general solution to the corresponding homogeneous equation.

(b) [9] Construct a particular solution to the non-homogeneous equation.

(c) [4] Find the solution with initial condition $y(0) = 2, y'(0) = 4$.

(d) [6] Use the Method of Undetermined Coefficients to write down the general form of the particular solution, y_p, for the differential equations given below but do **NOT** solve.

i. $y'' - 5y' + 4y = 3\cos(t)e^{4t}$

ii. $y'' - 5y' + 4y = t^2 e^{4t}$

iii. $y'' - 5y' + 4y = t^3$

Solution:

(a) Characteristic equation: $r^2 - 5r + 4 = 0$. Two roots: $r_1 = 1$ and $r_2 = 4$.
General solution to the homogeneous equation: $\boxed{y_h = C_1 e^t + C_2 e^{4t}}$.

(b) $2e^t$ is a homogeneous solution, therefore we must take $y_p = Ate^t$ with $y_p' = Ae^t + Ate^t$ and $y_p'' = Ae^t + Ae^t + Ate^t$. This gives $-3A = 3$. Therefore $A = -1$ and $\boxed{y_p = -te^t}$.

(c) The general solution is $y(t) = C_1 e^t + C_2 e^{4t} - te^t$. Applying initial conditions yields that

$$\begin{matrix} C_1 + C_2 = 2 \\ C_1 + 4C_2 - 1 = 2 \end{matrix}.$$

This gives $C_1 = C_2 = 1$. Therefore the solution to the IVP is $\boxed{y(t) = e^t + e^{4t} - te^t}$.

(d) The forms are as follows:

i. $\boxed{y_p = e^{4t}(A\cos(t) + B\sin(t))}$

ii. $\boxed{y_p = e^{4t}(At^2 + Bt + C)t}$

iii. $\boxed{y_p = At^3 + Bt^2 + Ct + D}$

7. [25] Short answer

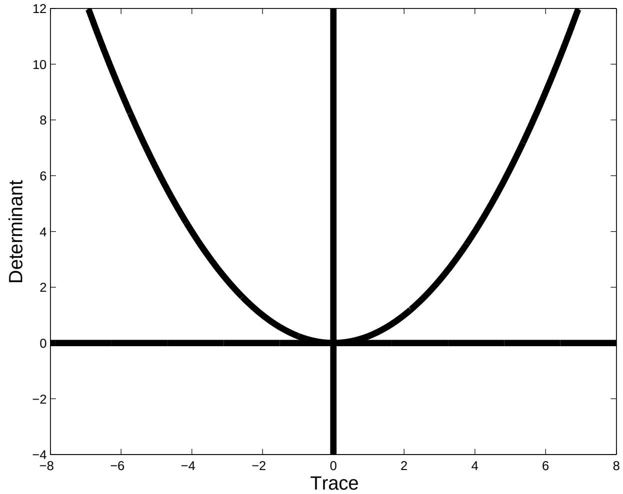
(a) [9] Which of the following systems have unstable equilibrium points at zero?

i. $y' = y$

ii. $y' = -2y + 2y^2$

iii. $y' = y(y-1)(y-2)$

(b) [16] Below is the trace/determinant plane for $\bar{x}' = A\bar{x}$ where A is any 2×2 matrix. For your convenience, the figure also contains curves that define the boundaries of behavior for different solutions.



Let A be the matrix

$$A(a) = \begin{bmatrix} a & 1 \\ a-2 & 1 \end{bmatrix}.$$

where two of the elements of A depend on the value of a parameter a. If a can be any real value, describe the different classifications that the steady state node $\bar{x}_* = (0, 0)$ can have.

Solution:

(a) [i and iii] Let $y' = f(y)$ be the generic equation.

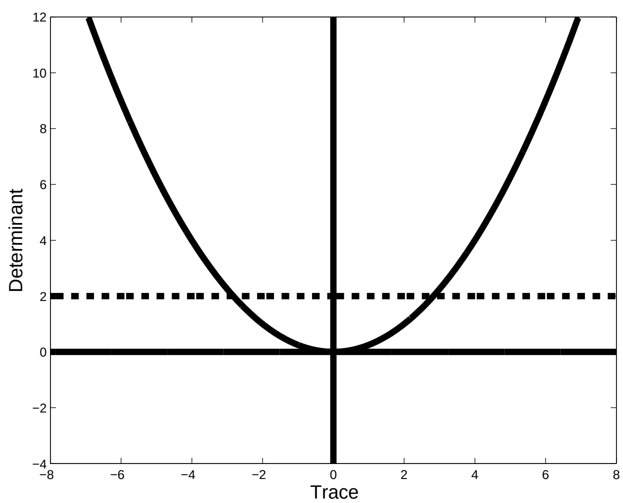
i. $f'(0) = 1$ and thus $y = 0$ is unstable.

ii. $f'(0) = -2$ and thus $y = 0$ is stable.

iii. $f'(0) = 2$ and thus $y = 0$ is unstable.

(b) [Attracting, repelling, degenerate, spirals (attracting and repelling), and center]

To solve this problem, we'll need to compute both the Trace and determinant of the matrix $A(a)$. The $\text{Tr}(A(a)) = a + 1$, while $|A(a)| = a - (a - 2) = 2$. Therefore, we need to graph the parametric plot $(x(a), y(a)) = (a + 1, 2)$ as a is varied



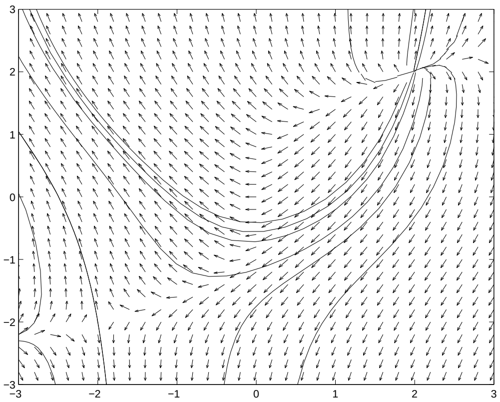
As is illustrated in the graph, the parametric curve is the dotted, horizontal line, intersecting the vertical axis at 2. Thus the node at (0,0) can be classified (moving from left to right along the dotted line) as attracting/sink, degenerate/star, attracting spiral, center, repelling spiral, degenerate/star, and repelling/source.

8. [24] Match each system with its phase portrait below. (You don't need to show your work.)

$$1.\left\{\begin{array}{lcl} x' &=& x+y \\ y' &=& -2x-y \end{array}\right. \qquad 2.\left\{\begin{array}{lcl} x' &=& -x+5y \\ y' &=& x-3y \end{array}\right. \qquad 3.\left\{\begin{array}{lcl} x' &=& y \\ y' &=& -x\sin(x) \end{array}\right.$$

$$4.\left\{\begin{array}{lcl} x' &=& x\sin(y) \\ y' &=& y\cos(x) \end{array}\right. \qquad 5.\left\{\begin{array}{lcl} x' &=& xy-4 \\ y' &=& y^3-4x \end{array}\right. \qquad 6.\left\{\begin{array}{lcl} x' &=& y-x \\ y' &=& x^2+y^2-8 \end{array}\right.$$

(A)



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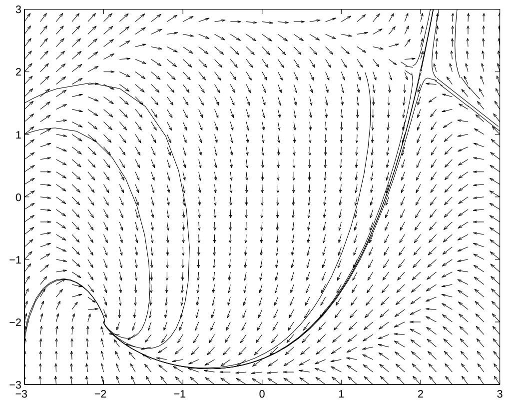

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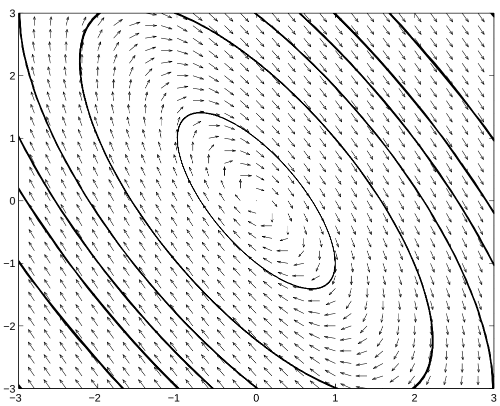

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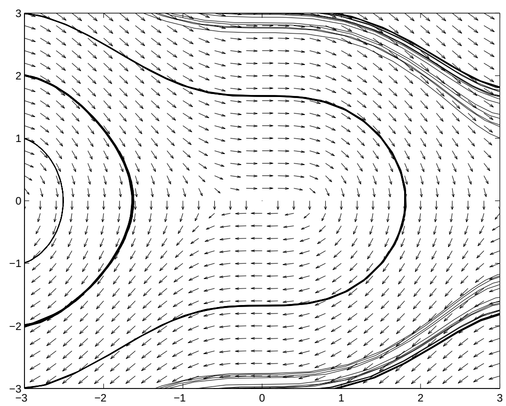

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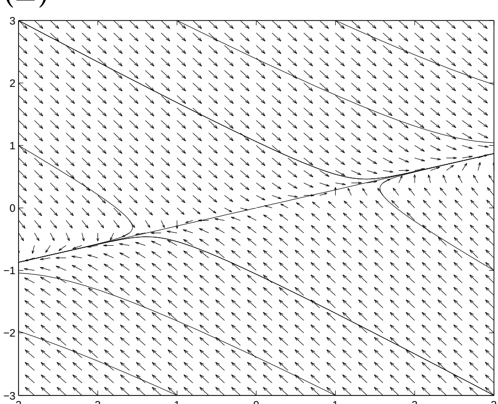
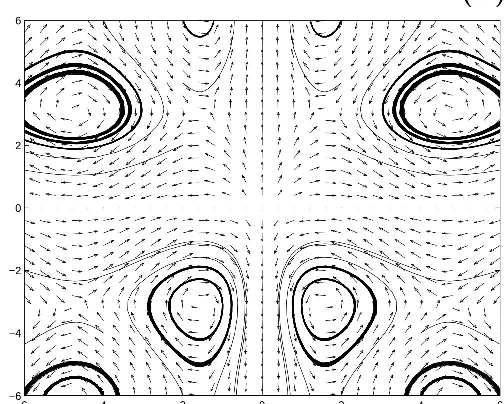


(F)



Solution: 1C, 2E, 3D, 4F, 5A, 6B