SOLUTION OF LINEAR SYSTEMS OF ALGEBRAIC EQUATIONS BY GAUSSIAN ELIMINATION

Gaussian elimination is the most practical way to solve linear systems of equations. Assuming we have equally many equations as unknowns, we can write the system

$$\left\{\begin{array}{ll} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = b_2 \\ \qquad\qquad \vdots & \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n & = b_n \end{array}\right. \tag{1}$$

Here, the *a*'s and the *b*'s are assumed to be known, and the task is to determine the values for the *x*'s. There are a few ways in which we can manipulate a system like this without affecting its solution:

1. Interchange the order of any two equations,
2. Multiply an equation by some constant,
3. Add a multiple of one equation to another.

The idea is to use these operations in a systematic manner, so that our system changes from the form (1) into

$$\left\{\begin{array}{rl} \alpha_{11}x_1 + \alpha_{12}x_2 + \ldots + \alpha_{1n}x_n & = \beta_1 \\ \alpha_{12}x_2 + \ldots + \alpha_{1n}x_n & = \beta_2 \\ \ddots \quad \vdots & \quad \vdots \\ \alpha_{nn}x_n & = \beta_n \end{array}\right. \tag{2}$$

Now we can, from the last equation, read off the value of x_n ; then from the next-to-last equation the value for x_{n-1} , etc. Finally, the top equation gives us our last unknown, x_1.

In matrix × vector form, we can write the original system (1) as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

We save even more writing if we don't repeat the *x*-vector at each step, so we adopt the even more compact notation

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right] \tag{3}$$

The three rules above for what we are allowed to do to a linear system of equations (without affecting its solution) become in this latest notation:

1. Interchange any two rows,
2. Multiply a row by a constant,
3. Add a multiple of a row to another row.

We want to apply these rules in such a way that (3) becomes upper triangular. As just observed, we can then solve for our unknowns, starting with the last one.

Example 1: Solve $\left\{\begin{array}{ll} 2x_1 - \ x_2 + x_3 & = 3 \\ x_1 + \ x_2 - x_3 & = 0 \\ 3x_1 + 2x_2 & = 7 \end{array}\right.$.

First (just in order to save on the writing), we put the system in compact form. We then drive it to upper triangular form. The strategy is to get one *column* at a time (from left to right) into the form we want. Subsequent row operations will then not damage any of the zeros we have already introduced:

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 0 \\ 3 & 2 & 0 & 7 \end{array}\right] \sim \qquad \left\{\begin{array}{l} \text{to make the algebra} \\ \text{nicer, swap row 1} \\ \text{and row 2} \end{array}\right\}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 2 & -1 & 1 & 3 \\ 3 & 2 & 0 & 7 \end{array}\right] \sim \qquad \left\{\begin{array}{l} \text{add -2} \times \text{row 1 to row 2} \\ \text{add -3} \times \text{row 1 to row 3} \\ \text{(first column then finished)} \end{array}\right\}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & -3 & 3 & 3 \\ 0 & -1 & 3 & 7 \end{array}\right] \sim \qquad \left\{\begin{array}{l} \text{need now to 'clear' column 2, so:} \\ \text{multiply row 2 by - } \tfrac{1}{3} \\ \text{after that, add row 2 to row 3} \end{array}\right\}$$

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 6 \end{array}\right] \qquad \left\{\begin{array}{l} \text{not necessary, but nicer: multiply row 3 by } \tfrac{1}{2} \\ \text{then re-write system in its explicit form} \end{array}\right\}$$

$$\left\{\begin{array}{rl} x_1 + x_2 - x_3 & = \ 0 \\ x_2 - x_3 & = -1 \\ x_3 & = \ 3 \end{array}\right. \qquad \left\{\begin{array}{l} \text{Back substitute:} \\ \text{use equations in} \\ \text{reverse order} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} x_3 = 3 \\ x_2 = -1 + x_3 = 2 \\ x_1 = -x_2 + x_3 = -2 + 3 = 1 \end{array}\right.$$

We can write the final solution as $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Example 2: Solve $\begin{cases} x_1 - x & = 1 \\ -x_1 + 3x_2 + 4x_3 & = 1 \\ x_1 - 3x_2 - 4x_3 & = -1 \end{cases}$

In compact form:

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ -1 & 3 & 4 & 1 \\ 1 & -3 & -4 & -1 \end{array}\right] \qquad \left\{\begin{array}{l} \text{add row 1 to row 2} \\ \text{add -1} \times \text{row 1 to row 3} \end{array}\right\}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & -2 & -4 & -2 \end{array}\right] \qquad \left\{\begin{array}{l} \text{add row 2 to row 3} \\ \text{then multiply row 2 by } \tfrac{1}{2} \end{array}\right\}$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left\{\begin{array}{l} \text{upper triangular form reached,} \\ \text{so re-write in explicit form,} \\ \text{back substitute} \end{array}\right\}$$

$$\begin{cases} x_1 - x_2 & = 1 \\ \quad x_2 + 2x_3 & = 1 \\ \qquad 0x_3 & = 0 \end{cases}$$

The last equation is satisfied by any value of x_3. So the best we can do is to set $x_3 = s$, where s is a free parameter. After that, back substitution proceeds as before:

$$\begin{cases} x_3 = s \\ x_2 = 1 - 2x_3 = 1 - 2s \\ x_1 = 1 + x_2 = 2 - 2s \end{cases}, \text{ or } \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}.$$

Example 3: Solve $\begin{cases} x_1 + 2x_2 & = 1 \\ x_1 + 2x_2 & = 2 \end{cases}$.

The compact form is $\left[\begin{array}{rr|r} 1 & 2 & 1 \\ 1 & 2 & 2 \end{array}\right]$ and, after adding (-1)×row 1 to row 2, we get $\left[\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & 0 & 1 \end{array}\right]$. This system is now in upper triangular form. The last equation tells that $0x_3 = 1$, so the system lacks a solution.

Although we started by assuming that we had a square coefficient matrix of the system (i.e. equally many equations as unknowns - the most usual case), there was no need for that assumption. Gaussian elimination works just as well (and leads to the complete set of solutions - if there are any) for all linear systems.