A DIFFUSIVE SYSTEM

1. Lab Objective

Solve a first-order linear system of differential equations which model diffusion across semi-permeable membranes. Then interpret the meaning of the result.

2. Instructions

This lab is due at the beginning of lecture on *Friday, December 8th, 2006.* You *must* also turn in *one* electronic copy per group to AMESS. Lab hours will be held in ECCR 143 from Friday, December 1st, 2006 to Thursday, December 7th, 2006.

You are strongly encouraged to work in groups of no more than three (3) people. If you work in a group, you need only hand in *one* lab per group. All labs must include a title page which includes *all* of the following information: *Name, Student I.D. Number, Professor, Recitation Number, and TA's Name.*

Remember, all your calculations (including those in your appendix) must be on printer paper. Additionally, the lab must be written in report form – that is, it should include an introduction, a body and a conclusion. Up to 20% of your grade may be based on organization, structure, style, grammar and spelling.

3. The Diffusive System

Consider three containers separated by a semi-permeable membrane, as shown in Figure 1. Assume that all the containers have equal volume and that they each contain some chemical in a solution with water. The boundaries between these containers are semi-permeable. That is, the chemical is allowed to pass through the membrane by the process of diffusion, but no water is allowed to pass between the containers. The membrane separating the first and second container is special in that it favors a concentration that is twice as high in container one than in container two. Further assume that the chemical is not saturated and remains in solution. We will investigate the diffusion of a chemical in this system using a system of first order differential equations.

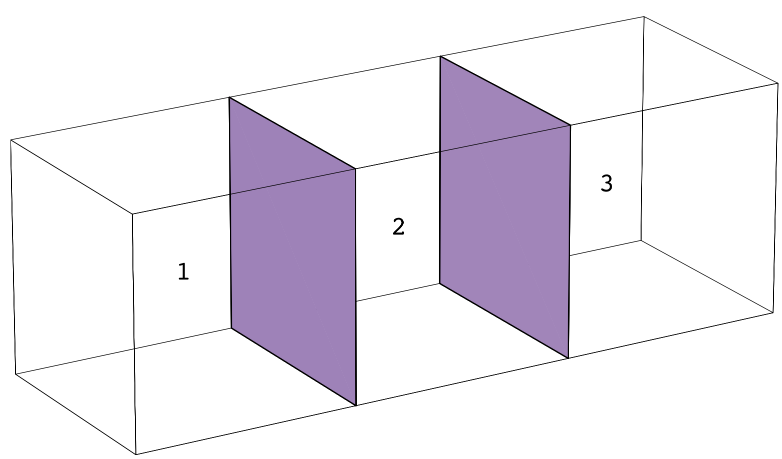

Figure 1. Three vats seperated by semi-permeable membranes.

3.1. **Derivation of the Model.** Using conservation laws, we can derive a system of differential equations to model the flow of the chemical through the membranes. The change in concentration of the chemical in one of the containers is proportional to the difference between the concentration of the adjacent containers. Let $y_1(t)$, $y_2(t)$, and $y_3(t)$ be the concentrations of the chemical in their respective containers. Using the conservation law and concentration defined by mass divided by volume of solution, we have the following system of differential equations:

$$\begin{aligned} y_1' &= k(-y_1 + 2y_2), \\ y_2' &= k(y_1 - 3y_2 + y_3), \\ y_3' &= k(y_2 - y_3), \end{aligned} \tag{1}$$

where the constant of proportionality, $k > 0$, depends upon the type of membrane.

4. Questions and Issues to Address

1.) Write the system in matrix form (see page 109 of section 3.1 in Farlow et al.). Call the coefficient matrix A, let $x = \begin{bmatrix} y_1(t) & y_2(t) & y3(t) \end{bmatrix}^T$. Classify the system using the terms on page 122 of section 3.1 in Farlow et al.

2.) Find all equilibrium solutions for this system, that is, solve $Ax = 0$, where A is the coefficient matrix and x is as in problem 1. Explain the physical meaning of the equilibrium solution. Is this system consistent or inconsistent? (For help see page 136 of Section 3.2 in Farlow et al.) Be sure to explain your answer.

3.) Calculate the determinant of A. You should do this by hand and show your work in the appendix, but you may refer to a software program to check your work. What does this say about the matrix A?

4.) For the remainder of this lab, set $k = 3$. Solve $\det(A - \lambda I) = 0$. To do this, first calculate the determinant. This will yield a cubic polynomial. Set the polynomial equal to zero and solve for the roots. These roots are called eigenvalues and are important for determining the analytical solution to the system.

5.) The next step in determining the analytical solution to this system is to find the corresponding eigenvectors. Eigenvectors are linearly independent vectors that make up one component of the analytical solution. In this case we must find three eigenvectors. To do this, we use Gauss-Jordan elimination to find the reduced row echelon form for three problems, (i.e., we solve $B\mathbf{v} = 0$ for three different coefficient matrices corresponding to the three eigenvalues determined in the previous problem). The system $B\mathbf{v} = 0$ can also be solved using Gaussian elimination with back substitution.

   (a) Determine the eigenvector (a column vector) $\mathbf{v}_1$ associated with one of your unique eigenvalues.

   (b) Be sure to clearly show the steps in Gauss-Jordan elimination or in Gaussian elimination with back substitution in the appendix.

   (c) Be sure to state which method you are using before solving, and include all calculations in the appendix – no credit will be given without the work.

   (d) Classify this system as consistent or inconsistent.

   (e) Choose one solution, that is, fix the arbitrary parameter in eigenvector, $\mathbf{v}_1$.

6.) Next find $\mathbf{v}_2$ is the same manner, but using a different eigenvalue. Follow the same guidelines as in the previous problem.

7.) One last time, determine $\mathbf{v}_3$ using your final eigenvalue. Again, follow the same guidelines as before.

8.) Determine if the three column vectors, $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$, are linearly independent or not. Be sure to discuss your methodology for testing for linear independence and include any necessary calculations.

9.) The general solution to this system of differential equations is given by the following:

$$\mathbf{y}(t) = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t} + c_3\mathbf{v}_3 e^{\lambda_3 t}, \tag{2}$$

where $\mathbf{y}(t) = \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) \end{bmatrix}^T$. Use this to determine the general solution to the system by substituting the eigenvectors, that is $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, you found in the previous problems. Note that, since there are infinitely many choices for each of the three eigenvectors, your results may not agree exactly with other students' results. Your choice depends on the scalar factor you chose when determining the column vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

10.) Use the initial condition $\mathbf{y}(0) = \begin{bmatrix} 1 & 2 & 4 \end{bmatrix}^T$ to determine the arbitrary constants c_1, c_2, c_3. What happens to $\mathbf{y}(t)$ as $t \to \infty$? What does this mean physically? What is happening to the concentration of the chemical in the containers? What would happen if we chose a different initial condition?

11.) Produce a graph of each of the functions $y_1(t), y_2(t)$, and $y_3(t)$ from the previous question (where $\mathbf{y}(t) = \begin{bmatrix} y_1(t) & y_2(t) & y_3(t) \end{bmatrix}^T$). Discuss the behavior for each function as $t \to \infty$. Be sure to
   (a) Label the graph and axes.
   (b) Choose an appropriate domain to adequately explore the function (make sure you don't miss anything!).

12.) Define $m(t) = y_1(t) + y_2(t) + y_3(t)$, where $y_1(t), y_2(t), y_3(t)$ are as defined in the previous question. Compute $m(t)$ and $m'(t)$.

13.) What are $m(t)$ and $m'(t)$? Give a physical interpretation for these quantities. Based on your discussion, are the results of your computation expected? Explain.

14.) In this analysis, we chose a specific value for the parameter k in order to carry out calculations. What is the physical meaning of k? What effect will changing the value of k have on the analysis we performed? Explain your reasoning.

15.) In our discussion of this diffusive system, we made several assumptions that simplified the model. For instance, we assumed that the semi-permeable membrane was the same between the containers and that only the chemical could pass through the membrane. Address any issues that you feel may improve the model. Be sure to discuss your assertions and provide justification for your claims.

References

[1] Written by Matt Nabity 10/2002 and slightly updated by Patrick Young 11/2006.