APPM 2360 - Project 2
Leontief Economic Models

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Abstract

Wassily Wassilyovitch Leontief won the Nobel Prize in 1973 for his work on how changes in one economic sector affect other economic sectors. His simple input-output model uses linear algebra extensively, and is therefore an excellent candidate for APPM 2360's Project 2. In this project, you will examine:

- A simple two-industry Leontief system.
- A more complex five-industry Leontief system.
- The existence and uniqueness of Leontief solutions.
- The effect of technology on Leontief solutions.

It is recommended that you read pages 152-154 of your textbook and that you familiarize yourself with Matlab or Mathematica to expedite your matrix computations! The course webpage for the 2360 lab course can be found at:

http://amath.colorado.edu/courses/2460/

1 A simple two-industry Leontief system

Suppose that we have two industries: electricity and transportation (E and T). The first step of the Leontief input-output analysis looks at the interdependency of these two industries on each other. For example, it is reasonable to assume that the production of electricity requires some electricity (to power the power plant) and some transportation (to bring coal or natural gas to the power plant). Similarly, it is reasonable to assume that the production of transportation requires both transportation (to supply parts or fuel) and electricity (to power manufacturing of transportation elements).

We call this **internal consumption** and we can place this information in a **technological matrix**, **A**, where $A_{X\to Y}$ is "the amount of X required to produce 1 of Y":

$$\mathbf{A} = \begin{pmatrix} A_{E\to E} & A_{E\to T} \\ A_{T\to E} & A_{T\to T} \end{pmatrix}$$

Now, suppose there is an external demand for electricity, d_E, and for transportation, d_T, which we can express in vector $\vec{d}$. The question that we attempt to answer is: *How much E and T should the industries produce to satisfy both demand and internal consumptions?* We will call this vector $\vec{x}$ with components x_E and x_T. In other words:

$$\vec{x} = \begin{pmatrix} x_E \\ x_T \end{pmatrix} \qquad \vec{d} = \begin{pmatrix} d_E \\ d_T \end{pmatrix}$$

The solution develops as follows:

$$\text{Total Output} = \text{External Demand} + \text{Internal Consumption}$$
$$\text{Then } \vec{x} = \vec{d} + \mathbf{A}\ \vec{x}$$

Task 1: Solve for $\vec{x}$ in terms of $\vec{d}$ and A. (Hint: your solution will also involve the identity matrix, I.

Now you will examine a specific example, and solve numerically. Suppose that for every \$1.00 of electricity produced, \$0.50 of electricity and \$0.30 of transportation are used. Similarly, for every \$1.00 of transportation produced, \$0.09 of transportation and \$0.30 of electricity are required.

Task 2: Write the matrix A for this system. Be sure to label your matrix

The world that surrounds this example industry demands \$500 of Electricity and \$600 of Transportation.

Task 3: How much Electricity and Transportation should be produced?

2 A more complex five-industry Leontief system

We now expand on the simple concept and examine a system that involves 5 industries. Suppose that there are 5 dimensions to our state space: Electricity, Trucks, Widgets, Computers, and Human Power.

- The electric company sees that it takes 0.15 units of electricity to make 1 unit of trucks; 0.23 units of electricity to make 1 widget; 0.23 units of electricity to make 1 unit of computers; 0.05 units of electricity to provide 1 unit of human power. It also takes 0.10 units of electricity to produce 1 unit of electricity, due to internal consumption.
- The truck company sees that it takes 0.10 units of trucks to produce 1 unit of electricity; 0.20 units of trucks to make 1 widget; 0.01 units of trucks to produce 1 unit of computers; 0.10 units of trucks to provide 1 unit of human power. It also takes 0.20 units of trucks to produce another unit of trucks.

- The tires company sees that it takes 0.05 widgets to produce 1 unit of electricity; 0.25 widgets to produce 1 unit of trucks; 0.01 widgets to produce 1 unit of computers; 0.10 widgets to provide 1 unit of human power. It also takes 0.05 widgets to produce another widget.
- The computer company sees that it takes 0.02 units of computers to provide 1 unit of electricity; 0.17 units of computers to produce 1 unit of trucks; 0.10 units of computers to produce 1 widget. It also takes 0.13 units of computers to provide another unit of computers, and furthermore, computers are not involved in the provision of human power at all.
- The HR staffing company sees that it takes 0.15 hundreds of employees to provide 1 unit of electricity; 0.20 hundreds of employees to provide 1 unit of trucks; 0.15 hundred employees to provide 1 widget; 0.10 hundred employees to produce 1 unit of computers. Also, the HR staffing company employs 0.15 people per person that it staffs out to other industries.

Task 4: Write out the A matrix for this industry. Be sure to label your matrix so it is well understood which rows and columns correspond to which industries.

Suppose that the World demands 1000 units of electricity, 2000 units of trucks, 3000 widgets, 4000 units of computers, and 500,000 human employees.

Task 5: How much of each should be produced to satisfy this demand?

Task 6: Could these industries satisfy *any* amount of demand that the world required? Why or why not?

3 The existence and uniqueness of Leontief solutions

For this section, suppose that our industrial system consists again (as in Part I) of only two industries: electricity and transportation. Suppose that for every $1.00 of electricity produced, $0.50 of electricity and $0.30 of transportation are used. Similarly, for every $1.00 of transportation produced, $0.82 of transportation and $0.30 of electricity are required.

Task 7: Write out the A matrix for this system. Again, do not forget labels.

Suppose there exist *two* countries, each with the industrial setup that you found above, in Task 7. In one country, the King of Rysztrepostan demands $10,000 of electricity and $20,000 of transporation. In the other country, the Duke of Meissenburgh demands $100,000 of electricity but donates $60,000 in transportation.

Task 8: Can the demands of the King of Rysztrepostan be met? Can

the demands of the Duke of Meissenburgh be met? Explain your conclusions here using *the power of linear algebra*!! Hint: attempt to make your explanation understandable to both the Count of Kaslovania, who understands economics but doesn't know math, and the Mayor of Larremopolis, who is clueless about economics but loves a good math text.

4 The effect of technology on Leontief solutions

In this section, we start to examine the effects of technology. We accept as our premise that technological development happens *and* that technology improves the efficiency with which one element of an economy can be converted into another.

Here's the story. Say there are two industries, X and Y. The first time that the economy was observed, it took 0.3 units of X to make one unit of X, and 0.3 units of Y to make one unit of X. Similarly, it took 0.2 Units of X to make one unit of Y, and 0.1 units of Y to make one unit of Y. However, economists continued to observe this economy, and found that due to technological increases, the amount of X it takes to produce one unit of Y is actually a decreasing function of time. The relationship they found was:

$$Price(t) \;=\; \frac{Price(0)}{Technology(t)}$$

where $Technology(t)$ could be called $\alpha(t)$ for the production of Y using X.

What's more, the economists noticed an interesting trend: the growth rate of technology α, with respect to time, was proportional to the current technology level α and furthermore, doubled every 10 years. This is a realistic model, in fact. For more information on an actual trend in technology development, read up on Moore's Law.

Furthermore, the economists noticed that the amount of Y used to produce X was *also* a function of time, due to a seasonal dependence on the number of daylight hours in a day (also realistic in agricultural markets). The relationship they found was:

$$Price(t) = 0.3cos(\frac{2\pi}{365}t)$$

(t in days)

Task 9: What is the A matrix for this economy when the economists first observed it? Don't forget labels.

Task 10: What is the A(t) matrix for this economy, that holds for any time, t? Don't forget labels.

Task 11: Suppose that external demand is static (not changing with time), asking for 150 units of X and 175 units of Y. Plot the productions of X and Y with respect to time, and be sure to include labels.

Comment on your results.

Task 12: What happens when the system runs for a long time? (At least 50 years!) Be sure to justify this both graphically and using your matrix from Task 10.

5 Things to write about

Task 12: What is realistic about the Leontief models? What isn't?

Task 13: Give another application outside of economics, for which a similar setup might be involved? Explain...