Fitting Velocity Profiles
APPM 2360, Fall 2009
Lab 2

November 2, 2009

1 Instructions

In this lab we will measure the ability of various functions to approximate real world data. This lab is designed to introduce you to the concept of least squares approximation and boundary value problems. There are topics at the end of the lab which need to be addressed in the lab write-up.

Labs may be done in groups of 3 or less. One report must be turned in for each group. Labs must include each student's:

- name
- student number
- section number
- Professor's name
- TA's name

This lab is due on Thursdsay, November $5^{th}, 2009$. **Late labs will not be accepted.** A paper copy of your group's lab must be turned in at recitation and **one** member of your group must upload an electronic copy of your lab to SafeAssign via CULearn. Detailed instructions on electronic submission can be found here. Any group which does not submit an electronic copy will receive a grade of 0. Labs will be returned in recitation to the person whose name is first on the report. Lab regrades must be given to your recitation teacher no later than the week following the return of the labs.

Simply answering the lab questions will not earn you a good grade. To fare well, you will need to pay close attention to the formatting guidelines. As much as 20% of your lab grade is based on formatting. Any questions regarding the lab policies may be addressed to the lab course coordinator, Ted Galanthay.

2 Introduction

Scientists often need to match curves to experimental data in order to verify models and determine physical constants. This is made difficult by the fact that experimental data contains errors. The data points never fall exactly on the curve predicted by the model. Mathematical models might also neglect some phenomena that appear in the data. The result of these errors and model deficiencies is that the systems of equations used to fit curves to the data almost never have solutions. In this lab you will develop approximate solutions to such systems using the method of least squares. You will be using curve fitting to compare models and data for fluid flow through a circular pipe. Your goal will be to verify two models for the velocity profiles. A velocity profile describes the variation in velocity on a cross section of the flow. This information is useful because integrating the velocity profile at some location along the pipe gives the volumetric flowrate through the pipe.

3 Models

The flow through the pipe is modeled in cylindrical coordinates. A diagram showing the coordinate system is shown in Figure 1. The variable r is the distance from center of the pipe and z is the distance along the pipe. If we apply a pressure gradient a flow will develop through the pipe. For relatively low fluid velocities the fluid all flows in the same direction down the pipe. This is called laminar flow. At higher velocities, the flow becomes unstable and develops numerous random motions in addition to the flow down the pipe. This is called turbulent flow.

For this particular problem, the velocity profile, u(r), for the laminar case can be found by solving the equation

$$\frac{d}{dr}(r\frac{du}{dr}) = Pr. \tag{1}$$

In Equation (1), P is a constant that combines effects of pressure and the fluid's viscosity. The unknown quantity u is the velocity in the axial (z) direction. In this problem, u is the only component of the velocity and it is a function of r only. For boundary conditions we require $u(R) = 0$ and $u(0)$ to be finite. The first boundary condition describes the tendency of the fluid in direct contact with the wall to be stationary. The need for the second boundary condition will be seen later. For now, it is enough to realize that an infinite velocity is physically impossible.

Turbulent flow presents a much more difficult problem. It is common to develop models and take measurements that just consider the average velocity at any point. The exact form of the random motions at any given time is generally not needed. The actual models for even this simple problem are fairly complicated, so we will just consider an empirical model for the velocity profile. An empirical model is one that fits experimental data reasonably well. It is not derived from the conservation laws. A commonly used empirical model for turbulent flow in a circular pipe is the following:

$$\tilde{u} = K(1-\tilde{r})^{\alpha} \tag{2}$$

In Equation (2) $(\tilde{u}) = \frac{u}{u_c}$ and $\tilde{r} = \frac{r}{R}$ where $u_c = u(0)$. (Note that in our discussion of turbulent flows, the variables $u, u_c, \tilde{u}$ refer to the *time average* of the flow velocity.) Using $\tilde{u}$ and $\tilde{r}$ will reduce the number of terms in the equations and remove the need to deal with units. It is possible to derive a linear relationship from Equation (2) using a simple transformation. Taking the natural logarithm of both sides of Equation 2 yields

$$\ln(\tilde{u}) = \alpha \ln(1-\tilde{r}) + \ln(K). \tag{3}$$

To emphasize the linear nature of Equation (3) we introduce the variables $\mu = \ln(\tilde{u})$ and $\rho = \ln(1-\tilde{r})$ and write equation (3) as

$$\mu = \alpha\rho + \ln(K).$$

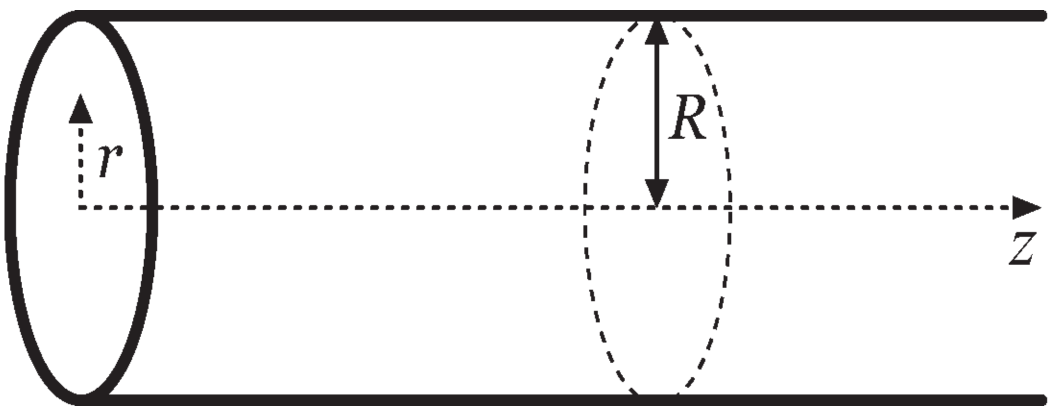

Figure 1: The problem discussed is for flow through a circular pipe with radius R. The relevant coordinates are the radial distance from the center of the pipe, r, and the distance along the length of the pipe, z.

4 Boundary Value Problems

The differential equation used to describe the laminar velocity profile in Equation (1) is a boundary value problem (BVP). There are two standard problems for ordinary differential equations (ODEs). In an initial value problem (IVP) all of the conditions used to determine the constants of integration are specified at one point. The problem will thus specify the function value and some number of derivatives at the starting point. In a boundary value problem the conditions for the integration constants are given at different locations. For a second-order ODE, the function value is specified at two points. As an example, we consider the BVP

$$\begin{aligned} u'' + u &= 0, \\ u(0) &= 2, \\ u(\frac{\pi}{2}) &= 1. \end{aligned} \tag{4}$$

Prove that the general solution to the equation $u'' + u = 0$ is $u(t) = c_1 \cos(t) + c_2 \sin(t)$ by directly substituting it into the equation. Apply the boundary conditions to find $u(0) = c_1 = 2$ and $u(\frac{\pi}{2}) = c_2 = 1$. Using the boundary conditions, we determine the constants c_1 and c_2 and obtain the solution $u(t) = 2\cos(t) + \sin(t)$. Try solving Equation (4) with $u(0) = 1$ and $u(\pi) = 2$. What happens? You do not need to discuss this example in your report, and it will not be graded.

5 Method of Least Squares

Consider the system of linear equations

$$A\vec{x} = \vec{b} \tag{5}$$

where $A = [a_{ij}]$ in an M x N matrix and $\vec{x} = [x_j]$ and $\vec{b} = [b_i]$ are column vectors with N and M elements respectively. The matrix A and the vector $\vec{b}$ are known while $\vec{x}$ is to be determined. We will be considering cases where $M \geq N$. There are many such cases when Equation (5) has no solutions due to experimental error, approximation, or various other causes. In such cases, one often looks for a vector, $\vec{x}$, that comes, in some sense, ”close” to satisfying Equation (5). The most common way of finding such a vector is the method of least squares. Rewriting Equation (5) gives

$A\vec{x} - \vec{b} = \vec{0}$. The problem here is that there is no $\vec{x}$ for which this is true, but minimizing the magnitude of $A\vec{x} - \vec{b}$ will yield an $\vec{x}$ that approximately satisfies Equation (5). Let F be the square of the magnitude of $A\vec{x} - \vec{b}$. We find that

$$F(\vec{x}) = |A\vec{x} - \vec{b}|^2 = \sum_{i=1}^{M}(\sum_{j=1}^{N} a_{ij}x_j - b_i)^2. \tag{6}$$

(We square the error $|A\vec{x} - \vec{b}|$ simply because it leads to simpler algebra.) The method is called "least squares" because it minimizes a sum of squared terms. At any critical point of F, we must have

$$\frac{\partial F}{\partial x_k} = 0. \tag{7}$$

It should be noted that Equation (7) represents N separate equations. The value of F is positive and can be made arbitrarily large, so if a single critical point is found it should be a minimum. Evaluating the derivative in (7), we obtain

$$\frac{\partial F}{\partial x_k} = \sum_{i=1}^{M} 2(\sum_{j=1}^{N} a_{ij}x_j - b_i)a_{ik} = 0. \tag{8}$$

After some algebra, Equation (8) yields

$$\sum_{i=1}^{M}\sum_{j=1}^{N} a_{ij}a_{ik}x_j = \sum_{i=1}^{M} a_{ik}b_i. \tag{9}$$

Equation (9) has the following convenient matrix form:

$$A^T A\vec{x} = A^T\vec{b}. \tag{10}$$

The N equations represented by Equation (10) are often called the normal equations for Equation (5). An important point here is that the normal equations *always* have at least one solution (and *only* one solution if A has rank N). We note that in the typical case, Equation (5) does not have a solution when $M > N$ which means that Equations (10) and (5) are typically not equivalent.

Remark: While the normal equations provide a mathematically simple formulation of the least squares problem, they are sometimes not ideal for scientific computations. The reason is technical and has to do with the way that round-off errors that arise in almost all computations that involve floating point numbers propagate through a computation. It turns out that the normal equations tend to magnify these round-off errors in a way that other more sophisticated algorithms do not. These matters are beyond the scope of this course and do not cause any difficulties for the problems that arise in this project.

6 Data

Experimental data has been collected showing the velocity profiles for a case with laminar and a case with turbulent flow. We will be comparing this data to the models to verify them and determine the constants for the turbulent model. The data collected is given in Table 1 for the laminar case and Table 2 for the turbulent case. All of the data has been non-dimensionalized using $\tilde{r} = \frac{r}{R}$ and $\tilde{u} = \frac{u}{u_c}$.

Table 1: Data for a laminar flow.

$\tilde{r}$	0.0	0.2	0.4	0.6	0.8	1.0
$\tilde{u}$	1.000	0.957	0.840	0.641	0.368	0.000

Table 2: Data for a turbulent flow.

$\tilde{r}$	0.0	0.2	0.4	0.6	0.8	1.0
$\tilde{u}$	1.000	0.984	0.921	0.878	0.789	0.000

7 Analysis

7.1 An Analytic Solution

1.) Solve Equation (1) subject to the boundary conditions $u(R) = 0$ and $u(0) = u_c < \infty$ (finite). This can be done with separation of variables. Integrate once to find $r\frac{du}{dr}$, and then separate again to find $u = u(r)$. When you apply the boundary conditions, the condition that requires $u(0)$ to be finite is satisfied by setting the integration constant of the term that causes a problem to be 0. The condition at $r = R$ should give you a value for the other constant.

2.) Calculate $u_c = u(0)$. Use $\tilde{r} = \frac{r}{R}$ and $\tilde{u} = \frac{u}{u_c}$ to substitute for r and u in your solution to Equation (1). You should get $\tilde{u}$ as a function of $\tilde{r}$ when you are done.

7.2 Fitting a Curve

3.) We will first try to fit the data in Table 1 with polynomials. Consider the linear, quadratic, and cubic polynomials as given in Equations (11) through (13).

$$\tilde{u} = c_0 + c_1\tilde{r} \tag{11}$$

$$\tilde{u} = c_0 + c_1\tilde{r} + c_2\tilde{r}^2 \tag{12}$$

$$\tilde{u} = c_0 + c_1\tilde{r} + c_2\tilde{r}^2 + c_3\tilde{r}^3 \tag{13}$$

For each of the polynomials, set up the system of equations that requires the curve to pass through all of the points in Table 1. You may place the full matrices in an appendix, but explain how you set them up in your report. What are the dimensions of the coefficient matrices? For each system determine if there is a unique solution, no solution, or more than one solution. Explain your reasoning.

4.) For each of the systems in question 3, construct the normal equations using Equation (10). What are the dimensions of the coefficient matrices? Calculate the determinants of each of these matrices. What does this tell you about the solutions of these systems? If possible, determine the solutions to each system and include the resulting polynomials in your report.

5.) On one plot, overlay the data and the 3 fitted polynomials. For the polynomials, show the full curves from $r = 0$ to $r = 1$, not just the points that match up with the data. Discuss how well these curves fit the data. Calculate $\sqrt{F}$ for each case using Equation (6). This value is called the residual. How does the residual relate to the quality of the polynomial fits observed in your plot?

6.) Compare your solution from questions 1 and 2 to the polynomials from question 4. Does the data support the model?

7.) Repeat questions 3 through 5 for the data in Table 2. You do not need to discuss matrix dimensions or determinants this time around. You should still discuss the residuals. You should find that your curves fit the data somewhat poorly. Why do you think this is the case?

7.3 Curve Fitting with Logarithms

8.) Let us try to improve on the result from the previous question. Convert the data in Table 2 to μ and ρ and plot it. (Note: You will need to drop the $\tilde{r} = 1$ data point from the plot and the rest of the calculations. This empirical model fails at the wall of the pipe.)

9.) Fit an appropriate polynomial (as per section 3) to the data and use this to calculate the value of the constants K and α in Equation (2). Add the plot of this polynomial to the plot of μ and ρ data points.

10.) Now, plot the untransformed data from Table 2 with the curve described in Equation (2). How does this fit compare to the data? Is Equation (2) an appropriate relationship for a curve fitting the velocity profile?

11.) In this lab, we have attempted to minimize the square of the magnitude of $|A\vec{x} - \vec{b}|$. There are two other commonly used methods to measure "closeness," the *infinity norm* and the *one norm*. The infinity norm measures closeness by examining the largest distance between the function value and the actual value. The one norm sums all the distances between the function value and the actual value. What are the infinity norm and one norm of the polynomial derived in problem 9? Compare this value with the least-squares error. What are the strengths and weaknesses of the three measures?