APPM 3010 - Review for Final Exam

1. Fundamentals
   - Given an ODE, possibly a higher order scalar equation and/or a nonautonomous equation, be able to put it in the form of a first order, autonomous system.
   - Given a multi-step map, be able to write it as a system of 1-step maps.
   - Be able to provide examples of various types of dynamical systems (e.g. maps and ODES, linear and nonlinear, autonomous and nonautonomous, high order scalar equations and systems of equations).
2. Well posedness
   - Be able to state the existence and uniqueness theorem.
   - Be able to state the theorems given in class regarding continuity in initial conditions and parameters.
   - Be able to explain why these ideas are important in modeling physical systems.
   - Know the difference between local and global existence.
   - Given an explicit $\mathbf{f}(\mathbf{x}, t, \lambda)$, be able to determine a range of initial conditions where the initial value problem (IVP) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t, \lambda)$ is locally well posed. Be able to determine a range of initial conditions where the corresponding solutions exist for all time.
   - Be able to give examples of IVPs where uniqueness fails.
   - Be able to give examples of IVPs whose solutions blow up in finite time.
3. Flows on the real line
   - Given $\dot{x} = f(x)$ either explicitly or as a graph, be able to determine the equilibria of the flow and their stability by graphing $f(x)$.

- Be able to determine the curvature of solutions by computing $\ddot{x} = f'(x)\dot{x}$.
- Given a phase portrait, be able to give an example of an IVP $\dot{x} = f(x)$ that is consistent with that phase portrait.
- Give a collection of solutions $x(t)$, be able to give an example of an IVP $\dot{x} = f(x)$ that is consistent with those solutions.
- Be able to determine the stability of an equilibrium point by linearization. From linearization, be able to determine the asymptotic rate of decay to or growth from the equilibrium point.

4. Flows on the circle

    - Be able to write down ODEs for a uniform and nonuniform oscillator.
    - Be able to describe physical problems that are modeled by a uniform and nonuniform oscillator.
    - Be able to identify the equilibrium points and periodic orbits for a given nonuniform oscillator $\dot{\theta} = f(\theta, a, \omega)$ by graphing $f(\theta, a, \omega)$ for different ranges of parameters. Be able to determine their stability by this analysis.
    - Be able to describe how a nonlinear oscillator behaves as the system approaches a parameter regime where equilibrium points appear (ghosts and bottlenecks).

5. Bifurcations

    - Given a first order equation with a parameter, be able to determine when, if at all, a bifurcation occurs.
    - Know the basic characteristics of the saddle-node, transcritical, and pitchfork bifurcations, including their bifurcation diagrams.
    - Be able to determine if one of these bifurcations occurs in a given system that is not in the canonical form.

6. One dimensional maps

    - Be able to find fixed points of simple maps explicitly and determine their stability using linear information.

- Be able to identify fixed points of more complicated maps graphically and determine their stability.
- Know the equation satisfied by a period - m orbit. Given the graph of the mth iteration map, be able to find period - m solutions. Know the stability criterion for period - m orbits and be able to apply it.
- Be able to describe in words the period doubling route to chaos.
- Be able to describe how the Lyapunov exponent can indicate a systems tendency towards chaos. **You need not memorize the definition of the Lyapunov exponent - it will be given if necessary.**
- Given the graph of a map's Lyapunov exponent versus a parameter, be able to indicate where the onset of chaos is expected and where intermittancy is expected.

7. Planar linear systems and linearization of planar nonlinear systems

   - Be able to reproduce Figure 5.2.8 on page 137 of Strogatz. Understand how the eigenvalues and eigenvectors of a matrix determine the local phase portrait of the corresponding dynamical system.
   - Know the difference between Lyapunov stability and asymptotic stability. Be able to provide an example of an equilibrium point that is Lyapunov stable but not asymptotically stable, and vice versa.
   - Know the difference between attracting and Lyapunov stable. Be able to provide and example of a fixed point that is attracting but not Lyapunov stable. Know that Lyapunov stable and attracting together imply asymptotically stable.
   - Know when the information from the linearization will predict the correct stability type of an equilibrium point under small nonlinear perturbations.
   - Know when the information from the linearization will predict the correct structure type of an equilibrium point under small nonlinear perturbations.

- Be able use information from the linearization to draw a rough picture of the local phase portrait for the corresponding nonlinear system.

8. Periodic orbits in planar systems

    - Know the two types of periodic solutions in a planar nonlinear system. Be able to give an example of each.
    - Be able to state the Poincare-Bendixson theorem and know how to apply it. Know the implications of this theorem in terms of chaotic behavior.
    - Given a nearly decoupled system in polar coordinates, be able to design a trapping region for the corresponding flow.
    - For more complicated flows, be able to show that a given region is a trapping region.
    - Know what nullclines are and how they are useful in determining the global flow.
    - Know what a Poincare map is and how it can be used to determine the existence and stability of a perioidic solution.
    - Know the basic characteristics of a Hopf bifurcation.

9. Hamiltonian systems

    - Be able to give examples of Hamiltonian system of ODEs
    - Given a smooth, scalar valued function of two variables, be able to write down the corresponding Hamiltonian system of differential equations.
    - Given a system of differential equations, be able to determine if it is a Hamiltonian system and if so, be able to find the Hamiltonian.
    - Be able to show that the Hamiltonian is a conserved quantity.
    - Be able to explain why the level curves of the Hamiltonian are tangent to the trajectories of solutions in the phase plane. Be able to draw the phase portrait using the level curves for the Hamiltonian.

- Be able to explain why isolated stable fixed points cannot be asymptotically stable in a Hamiltonian system.
- Be able to show that the linearization about a fixed point in a Hamiltonian system yields only saddles and centers. Understand how this linear information is related to the existence of special solutions (e.g. periodic solutions, homoclinic orbits, heteroclinic orbits) in this type of system.

10. Gradient systems

    - Be able to give examples of gradient system of ODEs.
    - Given a potential function, be able to write down the corresponding gradient system.
    - Given a system of ODEs, be able to determine whether or not it is a gradient system and if so, be able to determine its potential function.
    - Be able to explain why the level curves of the potential function are perpendicular to the flow of solution trajectories.
    - Be able to explain why gradient systems cannot have nontrivial periodic solutions.
    - Be able to explain why gradient systems cannot have stable fixed points that are not attracting.
    - Be able to explain why linearization of a gradient system will not predict spirals or centers.

11. Gradient-like systems

    - Know what a Lyapunov function is and why it is useful.
    - Given a mechanical system with friction, be able to come up with candidates for a Lyapunov function.
    - Be able to explain how the existence of a Lyapunov function can help rule out periodic solutions in a region of the phase plane.
    - Be able to explain how the existence of a Lyapunov function can help rule out fixed points that are stable but not attracting. Be able to provide an example of this phenomenon.

12. Chaos in 3-D continuous systems

   - Be able to show that solutions of the Lorenz system is invariant under the transformation $x \to -x$, $y \to -y$. Know what implications this has for the flow.
   - Be able to show that the Lorenz system is volume contracting. Know what implications this has for the possible asymptotic behavior of solutions.
   - Understand how to use a Lyapunov function to show that the fixed point at the origin is globally asymptotically stable for $r < 1$.
   - Be able to give a precise definition of chaos.
   - Be able to give a precise definition of an attractor.
   - Using these definitions, be able to explain why the Lorenz system exhibits chaos and possesses a strange attractor.
   - Be able to explain how the behavior of iterations for the Lorenz map gives evidence of chaos in the full system.
   - Be able to reproduce the bifurcation diagram for the Lorenz system (Strogatz, p. 330, fig. 9.5.1).