ANSWERKEY

APPM 3170: Discrete Applied Mathematics - Fall 2008

Exam #1
Lecturer: Manuel Lladser

INSTRUCTIONS: On the front of your bluebook please print your *name*, *student ID*, *course code*, *exam number*, *date* and *lecturer's name*. Please draw a grading table (with *2 columns* and *6 rows*). Show all your work in your bluebook. Please start each new problem in a *new page*. Solve the problems in the *same order* as they are requested. A correct answer with no supporting work may receive no credit while and incorrect answer with some correct work may receive partial credit. *Textbooks, class notes, graphing or programmable calculators, and crib sheets are not permitted.*

1. (20 points.) Imagine that you have been working as a cashier at a supermarket for sometime and that you would like to ask for a raise. To prepare your case to the store manager, it would be good to determine whom between Joseph, Robert, Lisa (your co-workers and friends) and yourself makes the most per hour. Based on your knowledge you know the following:

   - If Joseph doesn't make the most then neither does Rob
   - If Lisa doesn't make the most then Joseph does
   - If Rob doesn't make the most then you don't either

   (a) Rewrite the information provided as a compound proposition using the propositional variables Y="you make the most", J="Joseph makes the most", L="Lisa makes the most", and R="Rob makes the most".

   (b) If no two cashiers make the same per hour, what is the fewest number of your friends you would have to ask for their salary to determine who makes the most? What are their names? HINT: Consider using a truth table with only 5 rows.

2. (20 points.) A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *periodic* if there exists a certain $T > 0$ such that, regardless of the choice of $x \in \mathbb{R}$, $f(x) = f(x + T)$.

   (a) Rewrite the above definition as a mathematical proposition which uses quantifiers and f, T and x as propositional variables.

   (b) What is the negation of the above proposition?

   (c) Rewrite the proposition in part (b) in plain English.

   (d) Prove or disprove: $f(x) = x - \lfloor x \rfloor$ is periodic.

3. (20 points.)

   (a) Taking for granted that $\log_3(205665061800000) = 29.99...$, please respond: how many digits does the base-3 expansion of the number 205665061800000 have?

   (b) Determine the base-5 expansion of the number 1492.

(TWO MORE PROBLEMS ON THE BACK!)

4. (20 points.) Consider the following algorithm.

procedure $squares(n \geq 1\text{: integer})$
$i = 0$
$j = 1$
while $j^2 \leq n$ **do**
  $i = i + 1$
  $j = j + 1$
end while
$squares(n) = i$

Based on the above please respond:

(a) Compute $squares(10)$.

(b) What does $squares(n)$ represent? Provide a mathematical formula.

(c) Exactly how many sums are performed by the above algorithm to compute $squares(n)$?

(d) Show that the complexity of the above algorithm is $\Theta(n^{1/2})$.

5. (20 points.) Let S be a given set. The *power set* of S is the set defined as $\mathcal{P}(S) = \{A \mid A \subset S\}$ i.e. the power set of S is the set that contains all subsets of S as its elements.

(a) Show that $\mathcal{P}(S) \neq \emptyset$.

(b) Show that $|S| \leq |\mathcal{P}(S)|$.

(c) Show that for any function $f : S \to \mathcal{P}(S)$ the set

$$T = \{x \in S : x \notin f(x)\}$$

is not in the range of f.

(d) Use the above to conclude that S and $\mathcal{P}(S)$ cannot have the same cardinality.

DURATION: 90 MINUTES

Answer key Exam #1 - Fall 2008 - APPM 3170

P1 (a) $= (\sim J \Rightarrow \sim R) \wedge (\sim L \Rightarrow J) \wedge (\sim R \Rightarrow \sim Y)$

(b) : To say that "If Lisa doesn't make the most then Joseph does" is equivalent to say "Lisa or Joseph make the most". This is enough to respond the questions. However, in the possibility that some of the assumptions may be wrong, it is worth to construct a truth table.

J	R	L	Y	$\sim J \Rightarrow \sim R$	$\sim L \Rightarrow J$	$\sim R \Rightarrow \sim Y$	(a)
T	F	F	F	T	T	T	T
F	T	F	F	F	$\longrightarrow$	$\longrightarrow$	F
F	F	T	F	T	T	T	T
F	F	F	T	T	F	$\longrightarrow$	F

Hence only Joseph or Lisa make the most.

P2 (a) f is periodic $\equiv (\exists T>0)(\forall x \in \mathbb{R}): f(x) = f(x+T)$

(b) $= (\forall T>0)(\exists x \in \mathbb{R}): f(x) \neq f(x+T)$

(c) = for all $T>0$ there is an $x \in \mathbb{R}$ such that $f(x) \neq f(x+T)$

(d) : Let $x \in \mathbb{R}$. Observe that $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$, hence
$\lfloor x \rfloor + 1 \le x+1 < \lfloor x \rfloor + 2$ and therefore $\lfloor x+1 \rfloor = \lfloor x \rfloor + 1$
Therefore, $f(x+1) = x+1 - \lfloor x+1 \rfloor = x+1 - \lfloor x \rfloor - 1 = x - \lfloor x \rfloor = f(x)$.
This shows that f is periodic

P3 (a) : need $\lfloor 29.99\ldots \rfloor + 1 = 30$ digits.

(b) : $1492 = (21432)_5$ because

$$\begin{aligned} 1492 &= 298 \cdot 5 + 2 \\ &= 59 \cdot 5^2 + 3 \cdot 5 + 2 \\ &= 11 \cdot 5^3 + 4 \cdot 5^2 + 3 \cdot 5 + 2 \\ &= 2 \cdot 5^4 + 1 \cdot 5^3 + 4 \cdot 5^2 + 3 \cdot 5^1 + 2 \cdot 5^0 \end{aligned}$$

P4

(a) squares(10) = 3 b/c $1^2, 2^2, 3^2 \le 10$ however $4^2 > 10$

(b) squares(n) = $|\{j \ge 1 : j^2 \le n\}|$

(c) # sum's = 2 # while-cycles

$$= 2\,|\{j \ge 1 : j^2 \le n\}|$$

$$= 2\,|\{j \ge 1 : j \le \sqrt{n}\}| = 2\lfloor \sqrt{n} \rfloor .$$

(d) The complexity of the algorithm is $\Theta(\lfloor \sqrt{n} \rfloor)$.

But for all $x \ge 1$:

$$1 \le \frac{x}{\lfloor x \rfloor} \le \frac{\lfloor x \rfloor + 1}{\lfloor x \rfloor} = 1 + \frac{1}{\lfloor x \rfloor} \le 2$$

Hence

$$1 \le \frac{\sqrt{n}}{\lfloor \sqrt{n} \rfloor} \le 2, \text{ for all } n \ge 1 \text{ i.e. } \lfloor \sqrt{n} \rfloor = \Theta(\sqrt{n}).$$

Therefore, the algorithm has complexity $\Theta(\Theta(\sqrt{n}))$ i.e. $\Theta(\sqrt{n})$.

P5

(a) $\phi \in \mathcal{P}(S)$ hence $\mathcal{P}(S) \ne \phi$

(b) Define $g: S \to \mathcal{P}(S)$ as $g(x) = \{x\}$. If $x \in S$ then $\{x\} \subset S$ i.e. $g(x) \in \mathcal{P}(S)$. So g is well-defined. Finally, if $x_1 \ne x_2$ then $\{x_1\} \ne \{x_2\}$ i.e. $g(x_1) \ne g(x_2)$. This shows that g is one-to-one and hence $|S| \le |\mathcal{P}(S)|$.

(c) Suppose $\exists s \in S$ such that $f(s) = T$. If $s \notin f(s)$ then $s \in T = f(s)$ ※ If $s \in f(s) = T$ then $s \notin f(s)$ ※. Therefore, T cannot be in the range of f.

(d) By contradiction suppose that $|S| = |\mathcal{P}(S)|$ i.e. $\exists$ $f: S \to \mathcal{P}(S)$ that is bijective. Then f would be onto but this is not possible due to part (c). Hence such a function f cannot exist i.e. $|S| \ne |\mathcal{P}(S)|$.