ANSWER KEY

APPM 3170: Discrete Applied Mathematics - Fall 2008

Exam #2
Lecturer: Manuel Lladser

INSTRUCTIONS: On the front of your bluebook please print your *name*, *student ID*, *course code*, *exam number*, *date* and *lecturer's name*. Please draw a grading table (with *2 columns* and *6 rows*). Show all your work in your bluebook. Please start each new problem in a *new page*. Solve the problems in the *same order* as they are requested. A correct answer with no supporting work may receive no credit while and incorrect answer with some correct work may receive partial credit. *Textbooks, class notes, graphing or programmable calculators, and crib sheets are not permitted.*

1. (20 points) There are 64 counties in the State of Colorado.
   (a) Show that in any group of 200 Colorado residents at least 4 of them must reside in the same county.
   (b) What is the minimum number of Colorado applicants that need to apply to the CU undergraduate program to guarantee that at least 25 of them will be from the same county?

2. (20 points.)
   (i) Determine a (multiplicative) inverse of 12 modulo 5.
   (ii) Consider the linear congruence $(8x+7) = (11-4x) \pmod 5$. Solve this congruence applying only the following properties discussed in lecture. To receive full credit you must specify (by letter) each time you use one of the properties (listed below).
   (a) $a = a \pmod 5$.
   (b) If $a = b \pmod 5$ and $b = c \pmod 5$ then $a = c \pmod 5$.
   (c) If $a = b \pmod 5$ and $c = d \pmod 5$ then $(a + c) = (b + d) \pmod 5$.
   (d) If $a = b \pmod 5$ then $ac = bc \pmod 5$.

3. (20 points.) Consider a rectangular region of dimension $(2 \times n)$ and let a_n be the number of different ways in which this region may be tiled using tiles of dimensions (2×1), (1×2) and (2×2).
   (a) Determine a linear recursion with constant coefficients satisfied by a_n for $n \geq 2$.
   (b) Is this recursion homogeneous or non-homogeneous?
   (c) What is the characteristic equation of the recursion?
   (d) Determine a_0 and a_1.
   (e) Determine a_n explicitly for all $n \geq 0$.

4. (20 points.) Find an explicit particular solution to the recursion $a_n = 16a_{n-2} + n4^n$, with $n \geq 2$.

(ONE MORE PROBLEM ON THE BACK!)

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P1 (a) If there were no county with 4 or more residents, then at most 3 people would live in each county and hence the total # of people would be $\leq 3 \cdot 64 = 192$

$\gt\!\!\!\!\!*\!\!\!\!\!<$ Hence at least 4 of the residents must live in the same county.

(b) Need to find n such that $\left\lceil \frac{n}{64} \right\rceil \geq 25$ i.e.

$24 < \frac{n}{64}$. Hence, $\boxed{n = 24 \cdot 64 + 1}$

P2 (i) Notice that $2 \cdot 12 - 5 \cdot 5 = -1$. Hence $2 \cdot 12 \equiv -1 \pmod 5$

i.e. <u>(-2) is a multiplicative inverse of 12 modulo 5.</u>

(ii)

$$8x + 7 \equiv 11 - 4x \pmod 5$$
$$4x \equiv 4x \pmod 5 \quad \text{— by (a)}$$
$$12x + 7 \equiv 11 \pmod 5 \quad \text{— by (c)}$$
$$-7 \equiv -7 \pmod 5 \quad \text{— by (a)}$$
$$12x \equiv 4 \pmod 5 \quad \text{— by (c)}$$
$$* \quad (-2) \cdot 12x \equiv (-2) \cdot 4 \pmod 5 \quad \text{— by (d)}$$

But $(-2) \cdot 12 \equiv 1 \pmod 5$, hence

$$** \quad (-2) \cdot 12x \equiv x \pmod 5 \quad \text{— by (d)}$$

Now using $*$ and $**$, (b) implies $\boxed{x \equiv -8 \pmod 5}$

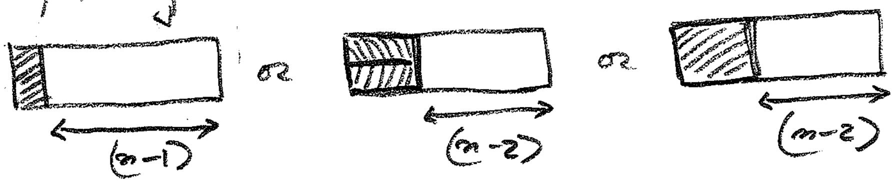
P4 The sols. to $r^2 = 16$ are $r = \pm 4$. Since $F(n) = n \cdot 4^n$, a particular sol. of the form $a_n = n(\alpha + \beta n) 4^n$ exists. To determine α and β observe that:

$$n(\alpha + \beta n) \cdot 4^n = 4^2 (n-2)(\alpha + \beta n - 2\beta) \cdot 4^{n-2} + n \cdot 4^n$$
$$\cancel{\alpha n} + \cancel{\beta n^2} = \cancel{\alpha n} - 2\alpha + \cancel{\beta n^2} - 2\beta n - 2\beta n + 4\beta + n$$
$$0 = (4\beta - 2\alpha) + (1 - 4\beta) n \text{, for all } n \geq 2$$

Hence $\beta = 1/4$, $\alpha = 1/2$ and $\boxed{a_n = n\left(\frac{1}{2} + \frac{n}{4}\right) \cdot 4^n = \frac{n}{2}\left(1 + \frac{n}{2}\right) 4^n}$

P3

(a) Any tilling with $n \geq 2$ must start as



Hence

$$a_n = 1 \cdot a_{n-1} + 1 \cdot a_{n-2} + 1 \cdot a_{n-2}$$

$$\boxed{a_n = a_{n-1} + 2a_{n-2}, \quad n \geq 2}$$

(b) $\boxed{\text{Homogeneous}}$

(c) $\boxed{r^2 = r + 2} \iff \boxed{r^2 - r - 2 = 0}$

(d) $\boxed{a_1 = 1}$ by definition. Since $a_2 = 2$ but also $a_2 = a_1 + 2a_0$ then must define $\boxed{a_0 = 1}$

(e) The solutions to (c) are $r = \dfrac{1 \pm \sqrt{1+8}}{2} = \dfrac{1 \pm 3}{2} = 2, -1.$

Hence

$$a_n = \alpha \cdot (-1)^n + \beta \cdot 2^n$$

Since

$$a_0 = \alpha + \beta \overset{\text{want}}{=} 1$$

$$a_1 = -\alpha + 2\beta \overset{\text{want}}{=} 1$$

then $\boxed{\beta = 2/3}$ and $\boxed{\alpha = 1/3}$

Therefore, $a_n = \dfrac{(-1)^n}{3} + \dfrac{2^{n+1}}{3} = \boxed{\dfrac{(-1)^n + 2^{n+1}}{3}, \quad n \geq 0}$

P5

Basic Step: $A^{0+1} = A = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & c \cdot \sum_{i=0}^{0} a^i b^{0-i} \\ 0 & b \end{bmatrix}$ ✓

Inductive Step: Assume the formula holds for n. Then

$$A^{(n+1)+1} = \begin{bmatrix} a^{n+1} & c \sum_{i=0}^{n} a^i b^{n-i} \\ 0 & b^{n+1} \end{bmatrix} \cdot \begin{bmatrix} a & c \\ 0 & b \end{bmatrix} = \begin{bmatrix} a^{n+2} & a^{n+1}c + c \sum_{i=0}^{n} a^i b^{n+1-i} \\ 0 & b^{n+2} \end{bmatrix}$$

But notice that

$$a^{n+1}c + c\sum_{i=0}^{n} a^i b^{n+1-i} = c\left\{\sum_{i=0}^{n} a^i b^{n+1-i} + a^{n+1} b^{(n+1)-(n+1)}\right\}$$

$$= c\sum_{i=0}^{n+1} a^i b^{n+1-i}$$

Therefore,

$$A^{(n+1)+1} = \begin{bmatrix} a^{n+2} & c\sum_{i=0}^{n+1} a^i b^{n+1-i} \\ 0 & b^{n+2} \end{bmatrix} \quad \checkmark$$

$\boxed{P6}$ Basic Step: $f(0) = a = \gcd(a,0)$ ✓

Inductive Step: Assume that $f(k) = \gcd(a,k)$, for all $0 \le k \le n$.

Need to show that $f(n+1) = \gcd(a, n+1)$.

Indeed, since $(n+1) \neq 0$, at the end of the first while-cycle,

$y = a \pmod{(n+1)}$ and $x = (n+1)$.