3310
Solutions
Exam 2

$\|v_1\| = \sqrt{3}$

$\|v_2\| = \sqrt{2}$

#1

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$\|v_3\| = \sqrt{6}$

$\|v_4\| = \sqrt{11}$

$$v_1 \cdot v_2 = 1\cdot 0 + (-1)(1) + (1)(1) + 0\cdot 0 = 0 - 1 + 1 + 0 = 0$$

$$v_2 \cdot v_3 = 0\cdot 2 + 1\cdot 1 + 1\cdot(-1) + 0\cdot 0 = 0 + 1 - 1 + 0 = 0$$

$$v_3 \cdot v_4 = 2\cdot 0 + 1\cdot 1 + (-1)(1) + 3\cdot 0 = 0 + 1 - 1 + 0 = 0$$

$$v_1 \cdot v_4 = 0 - 1 + 1 + 0 = 0$$

$$v_1 \cdot v_3 = 2 - 1 - 1 + 0 = 2 - 2 = 0$$

$$\boxed{v_2 \cdot v_4 = 0 + 1 + 1 + 0 = \boxed{2} \neq 0}$$

$$w_1 = r_{11} u_1$$

$$w_2 = \cancel{r_{12} u_1}^{0} + r_{22} u_2$$

$$w_3 = \cancel{r_{13} u_1}^{0} + r_{23} u_2 + r_{33} u_3$$

$$w_4 = \cancel{r_{14} u_1}^{0} + r_{24} u_2 + \cancel{r_{34} u_3}^{0} + r_{44} u_4$$

$$\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{u}_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

~~$\vec{u}_4 =$~~

$$r_{44} = \sqrt{11 - 0 - 2 - 0}$$

$$r_{44} = \sqrt{9} = 3$$

$$\vec{v}_4 = r_{24} \vec{u}_2 + r_{44} \vec{u}_4$$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \left\langle \begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} \right\rangle \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} + 3\vec{u}_1$$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \frac{2}{\sqrt{2}} \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} + 3 \left(\vec{u}_1 \right.$$

#1 continued

$$\begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + 3\vec{u}_2$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 3 \end{pmatrix} = 3\vec{u}_2$$

$$\boxed{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \vec{u}_2}$$

$$A = \underbrace{\begin{pmatrix} 1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 \\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{Q} \underbrace{\begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 2/\sqrt{2} \\ 0 & 0 & \sqrt{6} & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}}_{R}$$

Check:

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \quad \checkmark$$

Wow!

$$\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

2. $V = \text{span}\left\{ \underset{\vec{v}_1}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}} \right\}$ closest pt. to $\vec{b}$ in $V = ?$

a)

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$K = A^T A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\vec{f} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$K\vec{x} = \vec{f}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{aligned} 3x_1 &= 3 \qquad & x_1 &= 1 \\ 6x_2 &= 3 \qquad & x_2 &= \tfrac{1}{2} \end{aligned} \Bigg\} \text{ coords of } \vec{v}^*$$

$$\vec{x}^* = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \rightarrow \vec{v}^* = 1 \cdot \vec{v}_1 + \tfrac{1}{2}\vec{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} -1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 \\ -1/2 \\ 2 \end{pmatrix}$$

b) $d^* = \| \vec{v}^* - \vec{b} \| = \left\| \begin{pmatrix} 1/2 \\ -1/2 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\|$

$$= \left\| \begin{pmatrix} -1/2 \\ -1/2 \\ 0 \end{pmatrix} \right\| = \sqrt{\tfrac{1}{4} + \tfrac{1}{4}} = \sqrt{\tfrac{1}{2}} = \boxed{\tfrac{1}{\sqrt{2}}} = d^*$$

3. a) $K > 0$

K nonsingular $\Rightarrow$ Given $\vec{y} \in \mathbb{R}^n$ $\exists\, \vec{x} \in \mathbb{R}^n$ s.t. $K\vec{x} = \vec{y}$.

Then for $\vec{y} \in \mathbb{R}^n$

$$\begin{aligned} \vec{y}^T K^{-1} \vec{y} &= (K\vec{x})^T K^{-1} (K\vec{x}) \\ &= \vec{x}^T K^T K^{-1} K \vec{x} \end{aligned}$$

by $K = K^T$

$$\begin{aligned} &= \vec{x}^T \underbrace{K K^{-1}}_{I} K \vec{x} \\ &= \vec{x}^T K \vec{x} > 0 \end{aligned}$$

$$\Longrightarrow K^{-1} > 0. \blacksquare$$

b) Q_1, Q_2 $\perp$ matrices,

Then
$$\begin{aligned} Q^T Q &= (Q_1 Q_2)^T (Q_1 Q_2) \\ &= Q_2^T Q_1^T Q_1 Q_2 \\ &= Q_2^T Q_2 \\ &= I \blacksquare \end{aligned}$$

c) $K > 0$

Then
$$\begin{aligned} \vec{y}^T K^2 \vec{y} &= \vec{y}^T K K \vec{y} \\ &= \vec{y}^T K^T K \vec{y} \\ &= (K\vec{y})^T (K\vec{y}) \\ &= \vec{x}^T \vec{x} \\ &= \langle \vec{x}, \vec{x} \rangle > 0 \blacksquare \end{aligned}$$

Let $\vec{x} = K\vec{y}$

d) Ⓕ Counterexample

$K = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is NOT positive definite!
is not even regular.

e) $K > 0$ Let $\langle \ , \ \rangle$ be innerproduct on $\mathbb{R}^n$ defined by $\vec{x}^T K \vec{y}$

Then $\langle e_i, e_j \rangle = k_{ij}$ where $\{e_i\}_{i=1}^{n}$ standard basis vectors.

So $K = (\langle e_i, e_j \rangle)$

4. $\langle f, g \rangle = \int_{-1}^{1} f' g'$ inner product?

1) $\langle f, g \rangle = \int_{-1}^{1} f'g' = \int_{-1}^{1} g'f' = \langle g, f \rangle$ ✓

2) $\langle f, f \rangle = \int_{-1}^{1} f'f' = \int_{-1}^{1} (f')^2 > 0$ ✓

3) i) $\langle f+g, h \rangle = \int_{-1}^{1} (f+g)'h' = \int_{-1}^{1} (f'+g')h'$

$= \int_{-1}^{1} f'h' + g'h' = \int_{-1}^{1} f'h' + \int_{-1}^{1} g'h'$

$= \langle f, h \rangle + \langle g, h \rangle.$

ii) $\langle cf, g \rangle = \int_{-1}^{1} (cf)'g' = \int_{-1}^{1} c f'g = c\int_{-1}^{1} f'g' = c\langle f, g \rangle$

$= \int_{-1}^{1} f'(cg)' = \langle f, cg \rangle.$

5. Find Gram matrix K for $1, 2x, 3x^2$ under $L^2[0,1]$.

$$\langle 1, 1\rangle = \int_0^1 1 = \; = 1$$

$$\langle 1, 2x\rangle = \int_0^1 2x = x^2 \Big|_0^1 = 1$$

$$\langle 1, 3x^2\rangle = \int_0^1 3x^2 = x^3 \Big|_0^1 = 1$$

$$\langle 2x, 2x\rangle = \int_0^1 4x^2 = \frac{4x^3}{3}\Big|_0^1 = \frac{4}{3}$$

$$\langle 2x, 3x^2\rangle = \int_0^1 6x^3 = \frac{6x^4}{4}\Big|_0^1 = \frac{3}{2}$$

$$\langle 3x^2, 3x^2\rangle = \int_0^1 9x^4 = \frac{9x^5}{5}\Big|_0^1 = \frac{9}{5}$$

$$K = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 4/3 & 3/2 \\ 1 & 3/2 & 9/5 \end{pmatrix}$$

Yes, positive definite b/c $1, 2x, 3x^2$ are linearly independent vectors in $\mathcal{P}^{(2)}$.