Solution to the exam 1. 2009.

1. (a) False, $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

$AB = 0$, $BA = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \neq 0$.

(b) True, $\det(A) = \det(LU) = \det(L)\det(U) = \det(U)$

(c) F, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B$, $\det(A+B) = 4 \neq \det(A) + \det(B) = 2$.

(d) T, $AA^{-1} = A^{-1}A = I \Rightarrow (A^{-1})^T A^T = A^T (A^{-1})^T = I$
$\Rightarrow (A^T)^{-1} = (A^{-1})^T$.

(e) True, A singular $\Rightarrow$ rank$(A) < n$, $\Rightarrow$
there exists infinitely many solution
to $AX = 0$.

2 (a)
$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ -1 & -2 & 7 & 6 & b_2 \\ -1 & -2 & 2 & 2 & b_3 \end{array}\right) \xrightarrow[E_{31}(1)]{E_{21}(1)} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 10 & 10 & b_1+b_2 \\ 0 & 0 & 5 & 6 & b_1+b_3 \end{array}\right).$$

$$\xrightarrow{E_{32}(-\frac{1}{2})} \boxed{\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 10 & 10 & b_1+b_2 \\ 0 & 0 & 0 & 1 & b_3 + \frac{1}{2}(b_1-b_2) \end{array}\right)}.$$

(b) $\boxed{U = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 10 & 10 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$

$$L = E_{21}^{-1}(1)\, E_{31}^{-1}(1)\, E_{32}^{-1}(-\tfrac{1}{2}) = E_{21}(-1)\, E_{31}(-1)\, E_{32}(\tfrac{1}{2})$$

$$= \frac{E_{21}(-1)}{E_{31}(-1)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & \frac{1}{2} & 1 \end{pmatrix} = L}$$

$$A = LU$$

(c) $\operatorname{rank}(A) = 3$

(d) $\operatorname{rng}(A) = \{ \underset{m\times n}{A} x \mid x \in \mathbb{R}^n \}$

or $= \{ Ax \mid x \in \mathbb{R}^4 \}$ in here.

$\dim \operatorname{rng}(A) = 3 = m$, $\operatorname{rng}(A)$ has $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ as a basis or

using our method, the basis:

$\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 3 \\ 7 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}$ formed

by the pivoting columns of A.

P3

(e) $Ker(A) = \{ x \in \mathbb{R}^n \mid A_{m\times n} x = 0 \}$

or $Ker(A) = \{ x \in \mathbb{R}^4 \mid Ax = 0 \}$ here

x_2 is the free variable, let

$x_2 = 1$ and solve $Ux = 0$

we get $\boxed{v = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ is } \cancel{\text{the}} \text{ basis of } Ker(A)}$

(f) $rank(A) = 3 = m < n = 4$, always solvable with infinitely many solutions.

3. (a) A basis of a vector space V is a set of vectors $v_1, \dots v_n$ such that ⓐ $v_1, \dots v_n$ are linearly independent

ⓑ $V = span\{v_1, \dots v_n\}$

b) The set of continuous functions with $f(1)=0$ is closed under the operation $\alpha f + \beta g$ since if $f(1) = g(1) = 0$ then

$$(\alpha f + \beta g)(1) = \alpha f(1) + \beta g(1) = 0$$

The set of continuous functions with $f(1) = 1$ is not a subspace because for the function $f(x) \equiv 1$, $2f(1) = 2 \neq 1$, thus

P4

the set is not closed under multiplication by scalars.

(c) No, because $0_{m\times n}$ is not in $\mathbb{R}$.

(d) No, because

$$\sin(x+\tfrac{\pi}{3}) = \cos\tfrac{\pi}{3}\sin x + \sin\tfrac{\pi}{3}\cos x$$
$$= \tfrac{1}{2}\sin x + \tfrac{\sqrt{3}}{2}\cos x$$

or $\boxed{\tfrac{1}{2}\sin x + \tfrac{\sqrt{3}}{2}\cos x - \sin(x+\tfrac{\pi}{3}) = 0}$

(e) $V = \operatorname{Span}\{\sin x, \sin(x+\tfrac{\pi}{3}), \cos x\}$

find
$= \operatorname{Span}\{\sin x, \cos x\}$.

$c_1 \sin x + c_2 \cos x = 0 \Rightarrow c_1 = c_2 = 0$

$\Rightarrow \sin x$ and $\cos x$ are lin independ.

$\Rightarrow \dim V = 2$.

(f) $\sin x$ and $\cos x$ is a basis of V.