Exam II Fall 2009 Section 1

① a) $v_1 = w_1 = \begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$ $v_2 = w_2 - c v_1$ $c = \frac{-2}{2} = -1 \Rightarrow v_2 = \begin{pmatrix}0\\1\\0\\-1\end{pmatrix}$

orthonormal basis is $\boxed{(u_1, u_2) = \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right)}$

b) $\text{rank}(A) = 2$ by (a) so $\dim(\text{rng}(A)) = \dim(\text{corng}(A)) = \boxed{r = 2}$

$\dim(\ker(A)) = n - r = 4 - 2 = \boxed{2}$

$\dim(\text{coker}(A)) = m - r = 4 - 2 = \boxed{2}$

c) $\text{coker}(A) = (\text{rng}(A))^\perp$ so need $\begin{pmatrix}1 & 0 & -1 & 0\\ -1 & 1 & 1 & -1\end{pmatrix} \underset{\sim}{w} = 0$ for $w \in \text{coker}(A)$

$\Rightarrow w = c\begin{pmatrix}0\\1\\0\\1\end{pmatrix} + d\begin{pmatrix}1\\0\\1\\0\end{pmatrix} \Rightarrow \boxed{\text{coker}(A) = \text{span}\left(\begin{pmatrix}0\\1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}\right)}$

d) $b \in \text{rng}(A) = (\text{coker}(A))^\perp$

so if $y \in \text{coker}(A)$ then $y \cdot b = 0$ (Solvability conditions)

thus $w_1 \cdot b = 0 \Rightarrow \boxed{b_1 + b_3 = 0}$

$w_2 \cdot b = 0 \Rightarrow \boxed{b_2 + b_4 = 0}$

② a) Columns of $\bar{A}$ independent $\Rightarrow K = A^T A > 0$ (yes)

b) $A^T A$ is only semidefinite since 4 vectors in $\mathbb{R}^3$ cannot be independent

c) $A = \begin{pmatrix}1&0&1\\0&2&2\\1&2&5\end{pmatrix} \to \begin{pmatrix}1&0&1\\0&2&2\\0&2&4\end{pmatrix} \to \begin{pmatrix}1&0&1\\0&2&2\\0&0&2\end{pmatrix}$ since pivots > 0 and $A^T = A$ then $A > 0$

d) Not an inner product since not symmetric [it is bilinear & positive, however]

③ $K = A^T A = \begin{pmatrix}14 & 0\\ 0 & 31\end{pmatrix}$ (Since columns of A are $\perp$, K is diagonal

$f = A^T b = \begin{pmatrix}9\\4\end{pmatrix} \Rightarrow \underset{\sim}{x}^* = (A^T A)^{-1} A^T b = \boxed{\begin{pmatrix}9/14\\4/31\end{pmatrix}}$