Solution to Exam 2, 2009 Fall

1 (a) $V_1 = W_1 = (1,1,1,1)^T$, $V_2 = W_2 - \frac{\overset{=0}{\langle W_2, V_1\rangle}}{|V_1|^2} V_1 = W_2 = (1,0,-1,0)^T$,

$V_3 = W_3 - \frac{\langle W_3, V_1\rangle}{|V_1|^2} V_1 - \frac{\langle W_3, V_2\rangle}{|V_2|^2} V_2 = W_3 + V_2 = (0,1,0,-1)^T$

so $\boxed{u_1 = \begin{pmatrix} \frac12 \\ \frac12 \\ \frac12 \\ \frac12 \end{pmatrix}, \quad u_2 = \begin{pmatrix} \frac{1}{\sqrt2} \\ 0 \\ -\frac{1}{\sqrt2} \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt2} \\ 0 \\ -\frac{1}{\sqrt2} \end{pmatrix} \text{ is an ONB}}$

(b) From (a) $\dim \operatorname{rng} A = 3$, thus $\boxed{\dim \operatorname{corng} A = \operatorname{rank} = \dim \operatorname{rng} A = 3}$. $\boxed{\dim \ker A = 4-3 = 1}$, $\boxed{\dim \operatorname{coker} A = 4-3 = 1}$

(c) X is in coker A iff $X \perp V_i$, $i = 1,2,3$. or X solves $\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} X = 0$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & -2 & -2 \end{pmatrix} \qquad \boxed{V_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \text{ is the basis of coker} A}$$

(d) $AX = b$ solvable iff $b \perp \operatorname{coker} A$ iff $-b_1 + b_2 - b_3 + b_4 = 0$

For $b = (1,1,2,0)^T$, $-b_1 + b_2 - b_3 + b_4 = -2 \neq 0$ $\boxed{\text{not solvable}}$

For $b = (0,2,2,0)$ $-b_1 + b_2 - b_3 + b_4 = 0$ $\boxed{\text{solvable}}$

2 (a) $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$ ~~no sing~~ rank $= 2$.

$\Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \& \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are linearly independent so the Gram of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ are positive definite.

1b) $\dim(\mathbb{R}^4) = 4$, $6 > 4$, thus $v_1, \dots v_6$ must be linearly dependent $\Rightarrow$ the gram matrix is $\boxed{\text{not positive definite}}$

1c) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}$.

all pivots are positive $\Rightarrow$ The ~~matrix~~ symmetric matrix $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 9 \end{pmatrix}$ is positive definite.

1d) For $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\langle u, v \rangle = 1$, $\langle v, u \rangle = 0$

$\langle u, v \rangle \neq \langle v, u \rangle$ $\boxed{\text{so not an inner product}}$

3. (a) Find an orthogonal basis for W.

$v_1 = (1, 0, 3, 1)^T$, $v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = w_2 = (1, 1, 0, -1)$.

1b) Find the projection (which is also the closest point).

$$W = \frac{\langle b, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle b, v_2 \rangle}{\|v_2\|^2} v_2 = \frac{1}{3} v_1 = \boxed{\begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}}$$

$\boxed{(\tfrac{1}{3}, 0, \tfrac{2}{3}, \tfrac{1}{3}) \text{ is the closest point from } W \text{ to } b.}$