Homework Solutions: set 10

5.2.2 (a) $v_1 = w_1 = (1,0,1,0)^T$, $v_2 = w_2 - \frac{\overset{=0}{\langle w_2, v_1\rangle}}{\langle v_1, v_1\rangle} v_1 = (0,1,0,-1)^T$

$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{|v_1|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = w_3 - \frac12 v_1 + \frac12 v_2 = (\frac12, \frac12, -\frac12, \frac12)^T$

$v_4 = w_4 - \frac{\langle w_4, v_1\rangle}{|v_1|^2} v_1 - \frac{\langle w_4, v_2\rangle}{|v_2|^2} v_2 - \frac{\langle w_4, v_3\rangle}{|v_3|^2} v_3 = w_4 - \frac22 v_1 - \frac02 v_2 - \frac11 v_3 = w_4 - v_1 - v_3$
$= (-\frac12, \frac12, \frac12, \frac12)$.

$\boxed{u_1 = (\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}, 0)^T,\ u_2 = (0, \frac{1}{\sqrt2}, 0, \frac{-1}{\sqrt2})^T,\ u_3 = (\frac12, \frac12, -\frac12, \frac12)^T,\ u_4 = (-\frac12, \frac12, \frac12, \frac12)}$

5.2.7c $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & 2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$

$\text{rank}(A) = 3$, $\text{rng} A = \mathbb{R}^3$, with ONB $\boxed{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}}$;

$\text{Ker}(A) = \text{span}\left\{\begin{pmatrix}-1\\-1\\1\\1\end{pmatrix}\right\}$ with ONB: $\boxed{\begin{pmatrix}-\frac12\\-\frac12\\\frac12\\\frac12\end{pmatrix}}$

$\dim \text{coker} A = 3 - 3 = 0$, is a $\boxed{\text{trivial space}}$.

$\dim \text{corng} A = 3$, $w_1 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}$ $w_2 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ $w_3 = \begin{pmatrix}-1\\2\\0\\1\end{pmatrix}$ is a basis.

$v_1 = (1,0,1,0)^T$, $v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{|w_2|^2} v_1 = \begin{pmatrix}0\\1\\0\\1\end{pmatrix}$

$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{|v_1|^2} v_1 - \frac{\langle w_3, v_2\rangle}{|v_2|^2} v_2 = w_3 + \frac12 v_1 - \frac32 v_2 = (-\frac12, \frac12, \frac12, -\frac12)^T$

So $\boxed{u_1 = (\frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}, 0)^T,\ u_2 = (0, \frac{1}{\sqrt2}, 0, \frac{1}{\sqrt2}),\ u_3 = (-\frac12, \frac12, \frac12, -\frac12)^T}$ is an ONB of corng A.

5.2.17 b. $u_1 = \frac{1}{|w_1|} w_1 = \boxed{\frac{1}{\sqrt3}\begin{pmatrix}1\\1\\-1\\0\end{pmatrix}}$, with $r_{11} = \sqrt3$,

$r_{12} = \langle w_2, u_1\rangle = \frac{2}{\sqrt3}$, $r_{22} = \sqrt{|w_2|^2 - r_{12}^2} = \sqrt{\frac53}$

$u_2 = \frac{1}{r_{22}}(w_2 - r_{12}u_1) = \boxed{(\frac{-1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}}, \frac{3}{\sqrt{15}})^T}$

$r_{13} = \langle w_3, u_1\rangle = \frac{-1}{\sqrt3}$, $r_{23} = \langle w_3, u_2\rangle = \frac{1}{\sqrt{15}}$, $r_{33} = \sqrt{|w_3|^2 - r_{13}^2 - r_{23}^2} =$
$= \sqrt{\frac{48}{5}}$, $u_3 = \frac{1}{r_{33}}(w_3 - r_{13}u_1 - r_{23}u_2) = \boxed{(\frac{3}{\sqrt{15}}, \frac{-1}{\sqrt{15}}, \frac{2}{\sqrt{15}}, \frac{1}{\sqrt{15}})^T}$

5.3.1

(a) $A^TA = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \neq I$ $\boxed{\text{not orthogonal}}$.

(c) $A^TA = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\boxed{\text{orthogonal}}$

$\det A = 1 \det \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -1$ $\boxed{\text{improper}}$.

5.3.17: (a) $Q^TQ = (I - 2uu^T)(I - 2uu^T)$
$= I - 4uu^T + 4uu^Tuu^T$
$= I - 4uu^T + 4u(u^Tu)u^T$
$= I - 4uu^T + 4|u|^2uu^T = I - 4uu^T + 4uu^T = \boxed{I}$

(b)
$I - 2\begin{pmatrix} 1 \\ 0 \end{pmatrix}(1, 0) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$I - 2\begin{pmatrix} \frac{3}{5} \\ \frac{4}{5} \end{pmatrix}\left(\frac{3}{5}, \frac{4}{5}\right) = \begin{pmatrix} \frac{7}{25} & -\frac{24}{25} \\ -\frac{24}{25} & -\frac{7}{25} \end{pmatrix}$

$I - 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}(0, 1, 0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$I - 2\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(c) $Q^T = I - 2(uu^T)^T = I - 2(u^T)^Tu^T = I - 2uu^T = Q$.

(d) In general, $Qv = v - 2\underbrace{u(u^Tv)}_{\text{scalar}} = v - 2\underbrace{(u^Tv)}_{\text{scalar}}u = v$

$\Rightarrow (2u^Tv)u = 0 \qquad u \neq 0 \Rightarrow u^Tv = \boxed{\langle u, v\rangle = 0}$

5.3.27a $w_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $\quad r_{11} = \sqrt{5}$, $\quad u_1 = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix}$ P3

$w_2 = \begin{pmatrix}-3\\1\end{pmatrix}$, $\quad r_{12} = \langle w_2, u_1\rangle = \frac{-1}{\sqrt{5}}$,

$r_{22} = \sqrt{\|w_2\|^2 - r_{12}^2} = \frac{7}{\sqrt{5}}$, $\quad \boxed{R = \begin{pmatrix}\sqrt{5} & -\frac{1}{\sqrt{5}}\\ 0 & \frac{7}{\sqrt{5}}\end{pmatrix}}$

$u_2 = \frac{1}{r_{22}}(w_2 - r_{12}u_1) = \begin{pmatrix}-\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\end{pmatrix}$, $\quad \boxed{Q = \begin{pmatrix}\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}}\\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{pmatrix}}$

5.3.28(iii) $r_{11} = \sqrt{2}$, $\quad u_1 = \frac{1}{\sqrt{2}}(1,-1,0)$, $\quad r_{12} = \frac{1}{\sqrt{2}}$ $(\langle w_2, u_1\rangle)$

$r_{22} = \sqrt{2 - r_{12}^2} = \sqrt{\frac{3}{2}}$, $\quad u_2 = \frac{1}{r_{22}}\left(\begin{pmatrix}1\\0\\-1\end{pmatrix} - r_{12}\begin{pmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\\0\end{pmatrix}\right) = \begin{pmatrix}\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ -\frac{2}{\sqrt{6}}\end{pmatrix}$

$r_{13} = \langle w_3, u_1\rangle = -\frac{1}{\sqrt{2}}$, $\quad r_{23} = \frac{-1}{\sqrt{6}}$

$r_{33} = \sqrt{2 - \frac{1}{2} - \frac{1}{6}} = \frac{2}{\sqrt{3}}$, $\quad u_3 = \frac{1}{r_{33}}(w_3 - r_{13}u_1 - r_{23}u_2) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)^T$

$\boxed{R = \begin{pmatrix}\sqrt{2} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ 0 & \sqrt{\frac{3}{2}} & -\frac{1}{\sqrt{6}}\\ 0 & 0 & \frac{2}{\sqrt{3}}\end{pmatrix}}$ $\quad \boxed{Q = \begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}}\\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}}\end{pmatrix}}$

(b) $QRX = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$RX = Q^T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}}\end{pmatrix}$ $\quad \frac{2}{\sqrt{3}}x_3 = \frac{1}{\sqrt{3}} \Rightarrow x_3 = \frac{1}{2}$

$\sqrt{\frac{3}{2}}x_2 = \frac{1}{\sqrt{6}} + x_2\frac{1}{\sqrt{6}} = \frac{1}{2}\sqrt{\frac{3}{2}}$

$x_2 = \frac{1}{2}$, $\quad x_1 = \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}x_3 - \frac{1}{\sqrt{2}}x_2\right) = -\frac{1}{2}$

$\boxed{X = \begin{pmatrix}-\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\end{pmatrix}}$

P4

5.5.1 (b) $\langle v_1, \begin{pmatrix}2\\1\\1\end{pmatrix}\rangle = 3 \neq 0$ $\boxed{\text{not orthogonal}}$

$\langle v_2, \begin{pmatrix}1\\-1\\1\end{pmatrix}\rangle = -2-2+2 = -2 \neq 0$ $\boxed{\text{not orthogonal}}$

$\langle v_3, \begin{pmatrix}1\\-1\\1\end{pmatrix}\rangle = \langle v_3, \begin{pmatrix}2\\1\\1\end{pmatrix}\rangle = 0$ $\boxed{\text{Orthogonal}}$

$\langle v_4, \begin{pmatrix}1\\-1\\1\end{pmatrix}\rangle = 0$, $\langle v_4, \begin{pmatrix}2\\1\\1\end{pmatrix}\rangle = -2+3+4 = 5 \neq 0$ $\boxed{\text{not orthogonal}}$

(e) $\operatorname{rng} A = \operatorname{span}\left\{\begin{pmatrix}-3\\3\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}\right\}$

$\langle v_1, \begin{pmatrix}-3\\3\\-1\end{pmatrix}\rangle \neq 0$ $\boxed{\text{not orthogonal}}$

$\langle v_2, \begin{pmatrix}1\\-1\\0\end{pmatrix}\rangle \neq 0$ $\boxed{\text{not orthogonal}}$

$\langle v_3, \begin{pmatrix}1\\-1\\0\end{pmatrix}\rangle \neq 0$ $\boxed{\text{not orthogonal}}$

$\langle v_4, \begin{pmatrix}1\\-1\\0\end{pmatrix}\rangle \neq 0$ $\boxed{\text{not orthogonal}}$

5.5.3 $\langle \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}2\\-2\\2\end{pmatrix}\rangle = 0$, $v_1 = \begin{pmatrix}3\\2\\1\end{pmatrix}$, $v_2 = \begin{pmatrix}2\\-2\\2\end{pmatrix}$

$$u = \frac{\langle v, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle v, v_2\rangle}{\|v_2\|^2} v_2$$

$$= \frac{13}{14} v_1 + \frac{4}{12} v_2 = \begin{pmatrix}\frac{39}{14} + \frac{2}{3}\\ \frac{26}{14} - \frac{2}{3}\\ \frac{1}{14} + \frac{2}{3}\end{pmatrix} = \boxed{\begin{pmatrix}\frac{145}{42}\\ \frac{50}{42}\\ \frac{31}{42}\end{pmatrix}}$$

P5

5.5.9 ⓐ If $u, v \perp \overline{W}$, then for any scalar $\alpha, \beta \in \mathbb{R}$ and $w \in \overline{W}$, we have

$$\langle \alpha u + \beta v, w \rangle = \alpha \langle u, w \rangle + \beta \langle v, w \rangle = 0$$

$\Rightarrow \alpha u + \beta v \perp \overline{W} \Rightarrow \alpha u + \beta v$ also belong to the set. Thus the set of vectors that orthogonal to $\overline{W}$ is a subspace.

(b)
$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 2 & 0 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$$

$$\hookrightarrow \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -4 & 3 & 3 \end{pmatrix}$$

$\boxed{x_3, x_4 \text{ free variables}}$

$$\boxed{V_1 = \begin{pmatrix} -\frac{5}{2} \\ \frac{3}{4} \\ 0 \\ 1 \end{pmatrix} \qquad V_2 = \begin{pmatrix} -\frac{3}{2} \\ \frac{3}{4} \\ 1 \\ 0 \end{pmatrix}}$$

is a basis.