APPM 3310 EXAM #1

February 9, 2000

Name:

Read all **DIRECTIONS** carefully. Please show all work using extra paper if necessary.

1. Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ -2 & -3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 4 \\ 1 & -3 \\ 2 & 7 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & -3 \end{bmatrix}$.

   If possible, compute each of the following. If not possible, explain why. (15 pts.)

   a) $B^{\mathrm{T}} + 3D$ b) C^{2000} c) BC d) A^{-1}

   Answer:

   a)

$$\begin{bmatrix} 0 & 1 & 2 \\ 4 & -3 & 7 \end{bmatrix} + \begin{bmatrix} 3 & 12 & 15 \\ 0 & 6 & -9 \end{bmatrix} = \begin{bmatrix} 3 & 13 & 17 \\ 4 & 3 & -2 \end{bmatrix}$$

   b) Note that C is the elementary matrix that adds 3 times the first row to the third and -2 times the second row to the third. Iterating this procedure again and again gives

$$C^{2000} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 6000 & -4000 & 1 \end{bmatrix}$$

   c) Not possible. To compute BC, the number of columns in B (two) must be the same as the number of rows in C (three).

   d) Use the Gauss-Jordan method to compute the inverse of A.

$$\left[\begin{array}{ccc:ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ -2 & -3 & 2 & 0 & 0 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc:ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 2 & 0 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc:ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & -2 & 3 & -1 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

$$\Longrightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & 2 \\ 0 & 1 & 0 & -2 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \Longrightarrow A^{-1} = \begin{bmatrix} 3 & 1 & 2 \\ -2 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

   Note that you should check your answer by computing that $A^{-1}A = AA^{-1} = I$.

2. Suppose that F, G, L and O are all invertible matrices. What is $(FLOG)^{-1}$? (5 pts.)

Answer: $G^{-1}O^{-1}L^{-1}F^{-1}$ This is an extension of the rule $(AB)^{-1} = B^{-1}A^{-1}$.

3. **a)** Write down the 2×4 matrix A satisfying $a_{ij} = i^2 + 3j$. (5 pts.)

Answer:

$$A = \begin{bmatrix} 4 & 7 & 10 & 13 \\ 7 & 10 & 13 & 16 \end{bmatrix}$$

Recall that a_{ij} represents the entry of A in the i^{th} row, j^{th} column.

b) Write down a 2×2 skew-symmetric matrix B such that $B^2 = -9I$. (Recall that a skew-symmetric matrix satisfies $B^{\mathrm{T}} = -B$.) (5 pts.)

Answer: A 2×2 skew symmetric matrix is of the form

$$B = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix}$$

where b is any real number. Computing both sides of the equation $B^2 = -9I$ leads to $b^2 = 9$ so that $b = \pm 3$. There are then two possible answers

$$B = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \quad \text{or } B = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}.$$

4. The $(ij)^{\text{th}}$ entry of $A^{\mathrm{T}}B$ is (circle one only) (5 pts.)

a) The dot product of the i^{th} row of A with the j^{th} row of B.

b) The dot product of the j^{th} row of A with the j^{th} column of B.

c) The dot product of the j^{th} row of A with the i^{th} row of B.

d) The dot product of the i^{th} column of A with the j^{th} column of B.

e) The dot product of the i^{th} column of A with the j^{th} row of B.

Answer: d)

The $(ij)^{\text{th}}$ entry of $A^{\mathrm{T}}B$ is the dot product of the i^{th} row of A^{T} with the j^{th} column of B. But the i^{th} row of A_{T} is equivalent to the i^{th} column of A.

5. Suppose that A is a nonsingular $n \times n$ matrix. Circle **all** of the following which **must** be true? (Circle as many as you think are correct.) (10 pts.)

i) A is symmetric.

ii) The equation $Ax = b$ has a unique solution.

iii) There exists a matrix B such that $BA = I$.

iv) The determinant of A is zero.

v) If $AC = AD$, then $C = D$.

Answer: ii), iii), v)

If A is nonsingular, then A is invertible. A need not be symmetric at all. For example,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

is nonsingular but not symmetric. Choice **ii)** is the definition of nonsingular and choice **iii)** is part of the definition of being invertible. The determinant of A being **nonzero** is equivalent to A being nonsingular, so choice **iv)** is incorrect. Finally, choice **v)** follows from multiplying the equation on the left by A^{-1}.

$$A^{-1}(AC) = A^{-1}(AD) \Longrightarrow (A^{-1}A)C = (A^{-1}A)D \Longrightarrow IC = ID \Longrightarrow C = D.$$

6. Solve the given system by the method of Gaussian elimination. List the row operations you perform. What are the pivots?

$$\begin{aligned} 2u + v - 2w &= 1 \\ 4u + v + 3w &= 5 \\ -2u + 2v + w &= -10 \end{aligned}$$

What value for the coefficient of u in the third equation would force the process to break down completely? How many solutions are there in this case? (12 pts.)

Answer:

$$\begin{bmatrix} 2 & 1 & -2 & : & 1 \\ 4 & 1 & 3 & : & 5 \\ -2 & 2 & 1 & : & -10 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & 1 & -2 & : & 1 \\ 0 & -1 & 7 & : & 3 \\ -2 & 2 & 1 & : & -10 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 2 & 1 & -2 & : & 1 \\ 0 & -1 & 7 & : & 3 \\ 0 & 3 & -1 & : & -9 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & 1 & -2 & : & 1 \\ 0 & -1 & 7 & : & 3 \\ 0 & 0 & 20 & : & 0 \end{bmatrix}$$

Three steps: a) Row II - 2 x Row I, b) Row III + Row I, c) Row III + 3 x Row II

Pivots: $2, -1, 20$

Solving by back-substitution leads to $u = 2, v = -3, w = 0$.

For the second-half of the question, replace the -2 in the 31-th entry with an a and proceed, looking for a value which forces Gaussian elimination to break down.

$$\begin{bmatrix} 2 & 1 & -2 & : & 1 \\ 4 & 1 & 3 & : & 5 \\ a & 2 & 1 & : & -10 \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & 1 & -2 & : & 1 \\ 0 & -1 & 7 & : & 3 \\ a & 2 & 1 & : & -10 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 2 & 1 & -2 & : & 1 \\ 0 & -1 & 7 & : & 3 \\ 0 & 2 - \frac{a}{2} & 1 + a & : & -10 - \frac{a}{2} \end{bmatrix} \Longrightarrow \begin{bmatrix} 2 & 1 & -2 & : & 1 \\ 0 & -1 & 7 & : & 3 \\ 0 & 0 & 15 - \frac{5a}{2} & : & -4 - 2a \end{bmatrix}$$

The last pivot vanishes when $a = 6$. This forces Gaussian elimination to break down. The right-hand side's last entry is -16 when $a = 6$ so we obtain the equation $0 = -16$ which means there are no solutions for this case.

An easier way to see $a = 6$ is to add the first two equations. This produces $6u + 2v + w = 6$ which will be parallel (no solutions) with the bottom equation if the $-2u$ is replaced by $6u$.

7. For which value(s) of k does the system

$$\begin{aligned} x + ky &= 4 \\ kx + y &= 4 \end{aligned}$$

have (a) no solution, (b) one solution or (c) infinitely many solutions? (8 pts.)

Answer:

The easiest way to do the problem is look at the coefficient matrix

$$A = \begin{bmatrix} 1 & k \\ k & 1 \end{bmatrix}.$$

If the determinant of A is nonzero, then A is nonsingular and there is a unique solution. Since the determinant of A is $1-k^2$, the nonsingular case (part (b)) occurs when $k \neq \pm 1$. If $k = -1$, the equations become

$$\begin{aligned} x - y &= 4 \\ x - y &= -4 \end{aligned}$$

which has no solutions (part (a) - parallel lines). If $k = 1$, the equations become

$$\begin{aligned} x + y &= 4 \\ x + y &= 4 \end{aligned}$$

which has an infinite number of solutions (part (c) - same line).

8.

$$\text{Solve } LUx = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 8 \\ 17 \\ 11 \end{bmatrix}$$

without multiplying LU to find A. (8 pts.)

Answer:

First solve the triangular system $Lc = b$, which is,

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \begin{bmatrix} 8 \\ 17 \\ 11 \end{bmatrix}.$$

This yields $c_1 = 8, c_2 = -7, c_3 = 4$ using back-substitution.

Then solve $Ux = c$, which is

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ -7 \\ 4 \end{bmatrix}.$$

This yields $x_1 = 3, x_2 = 2, x_3 = 1$ using back-substitution.

9. Find the LDU decomposition of the matrix

$$A = \begin{bmatrix} 3 & 9 & -3 \\ 9 & 28 & -5 \\ -3 & -5 & 17 \end{bmatrix}. \qquad \text{(10 pts.)}$$

Answer:

Perform Gaussian elimination keeping track of the multipliers as you convert A to U.

$$\begin{bmatrix} 3 & 9 & -3 \\ 9 & 28 & -5 \\ -3 & -5 & 17 \end{bmatrix} \Longrightarrow \begin{bmatrix} 3 & 9 & -3 \\ 0 & 1 & 4 \\ -3 & -5 & 17 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 3 & 9 & -3 \\ 0 & 1 & 4 \\ 0 & 4 & 14 \end{bmatrix} \Longrightarrow \begin{bmatrix} 3 & 9 & -3 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \end{bmatrix}$$

The multipliers were $c_{21} = 3, c_{31} = -1, c_{32} = 4$. The pivots are $3, 1, -2$. To obtain the LDU factorization, the new U is found by dividing each row of the old U by its respective pivot. This gives

$$A = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that $U = L^{\mathrm{T}}$ which reflects the fact that A is symmetric.

10. What is the 4×4 permutation matrix which corresponds to interchanging the first and third rows with each other **and** interchanging the second and fourth rows with each other? (5 pts.)

Answer:

To find the correct permutation matrix, perform the indicated row exchanges on the 4×4 identity matrix. This yields

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Note that

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} c \\ d \\ a \\ b \end{bmatrix}$$

as desired.

11. Let A and B be any two n x n matrices. Is it true that $(A + B)(A - B) = A^2 - B^2$? Why or why not? (6 pts.)

Answer:

The statement is false as matrix multiplication is not usually commutative. Using the distributive property twice, we compute that

$$(A + B)(A - B) = A^2 - AB + BA - B^2.$$

This simplifies to $A^2 - B^2$ only when $AB = BA$.

12. Let A, B and C be n x n matrices. Suppose that C is invertible, $(A - B)C = 0$ and $B = A^{\mathrm{T}}$. Prove that A is symmetric. (6 pts.)

Answer:

We will use all three pieces of given information to show that $A = A^{\mathrm{T}}$. This is the definition of A being symmetric.

$$\begin{aligned}
(A - B)C = 0 &\implies AC - BC = 0 \\
&\implies AC = BC \\
&\implies AC(C^{-1}) = BC(C^{-1}) \\
&\implies A = B \\
&\implies A = A^{\mathrm{T}}
\end{aligned}$$

The fact that C is invertible is crucial to the proof. Note that we can have $AC = BC$, but **not** $A = B$. See homework problem # 4b in Section 1.6.