APPM 3310 EXAM #2

March 22, 2000

Name:

Read all **DIRECTIONS** carefully. Please show all work using extra paper if necessary.

A formula for free: The projection of $\mathbf{b}$ onto the line through $\mathbf{a}$ is

$$\mathbf{p} = \bar{x}\,\mathbf{a} = \frac{\mathbf{a}^{\mathrm{T}}\mathbf{b}}{\mathbf{a}^{\mathrm{T}}\mathbf{a}}\,\mathbf{a}$$

1. Which of the following subsets of $\mathbb{R}^2$ are subspaces? Circle **all** of the choices which you feel are correct. (10 pts)

i) The set of all vectors of the form $\begin{bmatrix} x \\ 0 \end{bmatrix}$ with $x \geq 0$.

ii) The set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x + y = 0$.

iii) The set of all vectors of the form $\begin{bmatrix} x \\ y \end{bmatrix}$ with $x + y = 2$.

iv) The set of all points on the curve $y = x^3$.

v) Span $\left\{ \begin{bmatrix} \sqrt{1.6} \\ -4\pi \end{bmatrix} \right\}$

Answer: ii, v

The only one dimensional subspace of $\mathbb{R}^2$ is a line through the origin. Choice **i)** is a half-line or ray. It fails to be closed under scalar multiplication (eg. $(2,0)$ is in the set but $-1 \cdot (2,0)$ is not in the set.) Choice **iv)** fails both closure properties. (eg. $(1,1)$ is in the set but $2 \cdot (1,1)$ is not in the set.) Choice **v)** is a subspace because the span of any set of vectors (dependent or independent) is **always** a subspace.

2. Suppose that we are given a set of nonzero vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ each in $\mathbb{R}^4$ such that $\mathbf{v_1} = -4\mathbf{v_2} + 5\mathbf{v_3} - 3\mathbf{v_4}$. Let $A = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}\ \mathbf{v_4}]$ be the matrix whose columns are the vectors in S. Circle **all** of the following choices which are true. (10 pts)

   i) S is a linearly dependent set.

   ii) A is invertible.

   iii) The rank of A is less than or equal to three.

   iv) The rows of A are linearly dependent.

   v) Every vector in $\mathbb{R}^4$ can be written as a linear combination of vectors in S.

   Answer: i, iii, iv

   Since $\mathbf{v_1}$ is multiple of the other vectors, the set S is linearly dependent. This means the columns of A are linearly dependent and so A is not invertible and must have rank less than or equal to three. (If A had rank 4, then it would be invertible.) By the Fundamental Theorem of Linear Algebra, row rank equals column rank and so the rows **must** also be linearly dependent. Since the 4 vectors in S are linearly dependent, they cannot form a basis for the four dimensional space $\mathbb{R}^4$, so choice **v)** is false.

3. Fill in the blank: $Ax = b$ has a solution if and only if b lies in the column space of A.

(4 pts)

Make up a system $Ax = b$ where A is a 3×2 matrix whose columns are linearly independent and b is a vector in $\mathbb{R}^3$, but there is no solution to $Ax = b$. (8 pts)

Answer: One easy solution is to choose the columns of A so that they span the xy-plane. Then choose b to be on the z-axis. Since Ax is in the column space and b is not, there will be no solution. (This was my reason for putting the fill in the blank question above.)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

gives a third equation of $0 = 1$ and thus no solution.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

also works to give a third equation of $0 = 1$ under Gaussian elimination.

4. Let $A = \begin{bmatrix} 1 & 0 & -2 & 3 & -2 \\ 2 & 0 & -6 & 6 & -4 \\ 1 & 0 & -4 & 3 & -2 \end{bmatrix}$

a) Find the rank of A.

b) Find a basis for the nullspace of A.

c) What is the dimension of the column space? Describe the column space geometrically. (ie. Is it a point? a line? a plane? the universe? What vector space does it live in?)

d) Find a basis for the left nullspace. Describe the left nullspace geometrically and relate it to the column space. (Whole problem is worth 20 pts)

First do Gaussian elimination on A to determine the number of pivots and obtain the reduced row-echelon form U.

$$A = \begin{bmatrix} 1 & 0 & -2 & 3 & -2 \\ 2 & 0 & -6 & 6 & -4 \\ 1 & 0 & -4 & 3 & -2 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \end{bmatrix} \Longrightarrow U = \begin{bmatrix} 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

a) Since there are two pivots (or two nonzero rows), the rank of A is two.

b) The pivots are in the first and third columns so the free variables are x_2, x_4, x_5 and the variables to be solved for are x_1 and x_3. Using $Ux = 0$ which has the same solution set as $Ax = 0$, we obtain the two equations

$$x_1 - 2x_3 + 3x_4 - 2x_5 = 0 \quad \text{and} - 2x_3 = 0.$$

This leads to

$$\begin{aligned} x_1 &= -3x_4 + 2x_5 \\ x_3 &= 0. \end{aligned}$$

Then,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_4 + 2x_5 \\ x_2 \\ 0 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and a basis for the nullspace of A is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

c) The dimension of the column space of A is equal to the rank of A which is 2. The column space lives in $\mathbb{R}^3$ and is therefore a plane passing through the origin.

d) Note that for the matrix A, the second row minus the first row equals the third row. Another way of saying this is

$$[1 \ -1 \ 1] \begin{bmatrix} 1 & 0 & -2 & 3 & -2 \\ 2 & 0 & -6 & 6 & -4 \\ 1 & 0 & -4 & 3 & -2 \end{bmatrix} = [0\,0\,0\,0\,0]$$

which means $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is in the left nullspace of A. Since we know the dimension of the left nullspace is $3 - 2 = 1$, this vector serves as a basis vector.

Alternatively, we can solve the system $A^{\mathrm{T}}x = 0$ which yields the equations $x_1 + 2x_2 + x_3 = 0$ and $-2x_2 - 2x_3 = 0$. Letting x_3 be the one free variable and solving for x_1 and x_2 in terms of x_3 gives $x_1 = x_3$ and $x_2 = -x_3$. This gives the same basis vector as above.

Since the left nullspace is orthogonal to the column space (Fund. Thm. of Lin. Alg. part 2), it is not only a line in $\mathbb{R}^3$, but the line **normal** to the plane determined by the column space. In other words, the equation for the plane of the column space must be $x - y + z = 0$ since it has $(1, -1, 1)$ as its normal vector.

5. Using the method of least squares, find the best straight line fit $b = C + Dt$ to the set of data $b = 1$ at $t = -2$, $b = 0$ at $t = -1$, $b = 4$ at $t = 0$, and $b = 3$ at $t = 3$. (14 pts)

Answer: There are four equations to be "solved".

$$\begin{aligned} C + D(-2) &= 1 \\ C + D(-1) &= 0 \\ C + D(0) &= 4 \\ C + D(3) &= 3 \end{aligned}$$

which is the system $Ax = b$ with

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 3 \end{bmatrix}, \quad x = \begin{bmatrix} C \\ D \end{bmatrix}, \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 3 \end{bmatrix}.$$

To solve using the method of least squares, we solve the system $A^{\mathrm{T}}A\bar{x} = A^{\mathrm{T}}b$. This yields

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \bar{C} \\ \bar{D} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \\ 3 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} 4 & 0 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} \bar{C} \\ \bar{D} \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}.$$

The solution is $\bar{C} = 2, \bar{D} = 1/2$ and the best straight line fit is the line

$$b = 2 + \frac{1}{2}t.$$

6. Suppose you have three vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ and you begin to apply the Gram-Schmidt process to obtain an orthonormal set.

a) Given that you have calculated $\mathbf{q_1}$ and $\mathbf{q_2}$, what is the formula(s) for obtaining the third orthonormal vector $\mathbf{q_3}$? (4 pts)

$$\mathbf{v_3}' = \mathbf{v_3} - (\mathbf{q_1}^{\mathrm{T}}\mathbf{v_3})\,\mathbf{q_1} - (\mathbf{q_2}^{\mathrm{T}}\mathbf{v_3})\,\mathbf{q_2}$$

and

$$\mathbf{q_3} = \frac{\mathbf{v_3}'}{||\mathbf{v_3}'||}$$

b) Use your formula from part **a)** to calculate $\mathbf{q_3}$ for the vectors

$$\mathbf{v_1} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v_3} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 4 \end{bmatrix}, \quad \mathbf{q_1} = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{q_2} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

(6 pts)

$$\mathbf{v_3}' = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 4 \end{bmatrix} - \frac{5}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 2 \end{bmatrix}.$$

Since $||\mathbf{v_3}'|| = \sqrt{13}$, we have

$$\mathbf{q_3} = \frac{1}{\sqrt{13}} \begin{bmatrix} 1 \\ 2 \\ -2 \\ 2 \end{bmatrix}.$$

c) If $Q = [\mathbf{q_1}\ \mathbf{q_2}\ \mathbf{q_3}]$ is the matrix whose columns are from the orthonormal set in part **b)**, what is $Q^{\mathrm{T}}Q$? Be precise in your answer. (4 pts)

$$\begin{bmatrix} \mathbf{q_1}^{\mathrm{T}} \\ \mathbf{q_2}^{\mathrm{T}} \\ \mathbf{q_3}^{\mathrm{T}} \end{bmatrix} [\mathbf{q_1}\ \mathbf{q_2}\ \mathbf{q_3}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

since

$$\mathbf{q_i}^{\mathrm{T}}\mathbf{q_j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

7. True or False. In each case, provide justification for your choice with a calculation and/or written argument. (20 pts)

Note: On average, these were the hardest questions on the test.

a) The vector space of 4×4 symmetric matrices has dimension 6.

FALSE

This space has dimension 10 as the number of free entries is equal to 10.

$$\begin{bmatrix} a & e & h & j \\ & b & f & i \\ & & c & g \\ & & & d \end{bmatrix}$$

b) There does **not** exist a matrix whose row space contains $(2, 1, 0)$ and whose null space contains $(0, 2, 1)$.

TRUE

This follows from the Fundamental Theorem of Linear Algebra part 2. For any matrix, the row space and nullspace are always orthogonal. However the vectors $(2, 1, 0)$ and $(0, 2, 1)$ are not orthogonal since their inner product is 2. Thus no such matrix can exist.

c) If x_1 and x_2 are each solutions to $Ax = b$, then $x_1 - x_2$ is in the nullspace of A.

TRUE

To see if $x_1 - x_2$ is in the nullspace of A compute $A(x_1 - x_2)$.

$$\begin{aligned} A(x_1 - x_2) &= Ax_1 - Ax_2 && \text{by linearity} \\ &= b - b && \text{by hypothesis} \\ &= 0. \end{aligned}$$

Therefore, $x_1 - x_2$ is in the nullspace of A. This question is actually a justification for why any solution to $Ax = b$ is of the form $x = x_p + x_h$.

d) The sum of two orthogonal $n \times n$ matrices is also an orthogonal matrix.

FALSE

If the columns of Q_1 and Q_2 each have unit length, then, in general, one would expect the sum $Q_1 + Q_2$ to have columns which do not have unit length.

Nice counterexample:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

While the first two matrices are orthogonal (they are the two 2×2 permutation matrices), their sum is clearly not (it is not even invertible!).