APPM 3310 EXAM #1 Solutions

February 9, 2001

1. Solve the given system by the method of Gaussian elimination, keeping track of the row operations you perform. What are the pivots? Let A be the 3×3 coefficient matrix for the left-hand side. Factor A into LU. (20 pts.)

$$\begin{array}{rcrcrcr} 2u & + & & + & w & = & -5 \\ -4u & + & v & & & = & 6 \\ 6u & + & 3v & + & w & = & -3 \end{array}$$

Solution: Use Gaussian elimination on the augmented matrix

$$\left[\begin{array}{rrrcr} 2 & 0 & 1 & : & -5 \\ -4 & 1 & 0 & : & 6 \\ 6 & 3 & 1 & : & -3 \end{array}\right] \Longrightarrow \left[\begin{array}{rrrcr} 2 & 0 & 1 & : & -5 \\ 0 & 1 & 2 & : & -4 \\ 0 & 3 & -2 & : & 12 \end{array}\right] \Longrightarrow \left[\begin{array}{rrrcr} 2 & 0 & 1 & : & -5 \\ 0 & 1 & 2 & : & -4 \\ 0 & 0 & -8 & : & 24 \end{array}\right]$$

The three row operations performed were:
a) Row II + 2 Row I
b) Row III - 3 Row I
c) Row III - 3 Row II

Solving by back-substitution yields, $w = -3, v = 2$ and $u = -1$.

The pivots are the diagonal entries of the upper triangular matrix obtained after Gaussian elimination: $2, 1, -8$.

Finally, the opposite of the multipliers obtained from the three row operations go into the lower-triangular matrix L.

$$L = \left[\begin{array}{rrr} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 3 & 1 \end{array}\right] \quad \text{and} \quad U = \left[\begin{array}{rrr} 2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{array}\right].$$

2. By using the Gauss-Jordan method, compute the inverse of

$$A = \left[\begin{array}{rrr} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right].$$

Also find the inverse of A^{T}. (15 pts.)

Solution: Apply Gaussian elimination on the augmented matrix $A : I$ to solve $AA^{-1} = I$ all at once.

$$\left[\begin{array}{rrrcrrr} 1 & 2 & 3 & : & 1 & 0 & 0 \\ 0 & 1 & 2 & : & 0 & 1 & 0 \\ 0 & 0 & 1 & : & 0 & 0 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{rrrcrrr} 1 & 0 & -1 & : & 1 & -2 & 0 \\ 0 & 1 & 0 & : & 0 & 1 & -2 \\ 0 & 0 & 1 & : & 0 & 0 & 1 \end{array}\right] \Longrightarrow \left[\begin{array}{rrrcrrr} 1 & 0 & 0 & : & 1 & -2 & 1 \\ 0 & 1 & 0 & : & 0 & 1 & -2 \\ 0 & 0 & 1 & : & 0 & 0 & 1 \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The three row operations performed were:
a) Row I - 2 Row II
b) Row II - 2 Row III
c) Row I + Row III

To find the inverse of A^{T} use the fact that $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$ and take the transpose of the answer just obtained.

$$(A^{\mathrm{T}})^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}.$$

Quickies

3. Suppose that A, C, E, T and U are all invertible matrices and that $UT = CAE$. Find an expression for A^{-1} ? (10 pts.)

   Solution: First multiply on the left of both sides by C^{-1} and on the right of both sides by E^{-1} to obtain an expression for A.

$$C^{-1}UTE^{-1} = (C^{-1}C)A(EE^{-1}) \Longrightarrow A = C^{-1}UTE^{-1}$$

   Then use the formula $(AB)^{-1} = B^{-1}A^{-1}$ to invert A:

$$A^{-1} = ET^{-1}U^{-1}C$$

4. What is the 3×3 permutation matrix which interchanges the first and second rows and then interchanges the second and third rows? (8 pts.)

   Solution: There are two ways to get the answer:

   1) Note that the operation sends

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \to \begin{pmatrix} b \\ a \\ c \end{pmatrix} \to \begin{pmatrix} b \\ c \\ a \end{pmatrix}.$$

   This tells us which columns to put the ones in in each row. The (12)th entry, the (23)rd entry and the (31)th entry.

   2) Multiply the matrices P_{12} and P_{23} together in the correct order. We want $P_{23}P_{12}$ because the first operation done is interchanging the first and second rows. It is on the right in the product because it will multiply a vector v first.

$$P_{23}P_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

5. The $(ij)^{\text{th}}$ entry of AB^{T} is **(Circle one only.)** (8pts.)

   i) the dot product of the i^{th} row of A with the j^{th} column of B.

   ii) the dot product of the i^{th} row of A with the j^{th} row of B.

   iii) the dot product of the i^{th} column of A with the j^{th} column of B.

   iv) the dot product of the i^{th} column of A with the j^{th} row of B.

   Solution: The answer is **ii)** because the $(ij)^{\text{th}}$ entry of AB^{T} is the dot product of the i^{th} row of A with the j^{th} column of B^{T}, but the j^{th} column of B^{T} is equal to the j^{th} row of B.

6. Suppose that A is an invertible $n \times n$ matrix. Circle **all** of the following which **must** be true? (**Circle as many as you think are correct.**) (12 pts.)

   i) A is singular.

   ii) The equation $Ax = b$ has a unique solution.

   iii) No two rows of A are identical.

   iv) The diagonal entries of A are all nonzero.

   Solution:

   Items **ii)** and **iii)** must be true. Item **i)** is false because if A is invertible than $Ax = b$ has a unique solution (not 0 or ∞ solutions). The solution is $x = A^{-1}b$. Item **iii)** is true because if two rows were identical, then Gaussian elimination would yield a row of zeroes and consequently no pivots, which contradicts the fact that an invertible matrix has a full set of nonzero pivots. Item **iv)** is false. It is the diagonal entries of U, the matrix obtained from A after Gaussian elimination, which must be zero. The permutation matrices, all of which are invertible, always have at least two zeroes on their diagonal. (Why?)

7. **True or False:** If the item is true, give a short proof or coherent mathematical argument to why it is true. If the item is false, explain why or provide a counterexample. (9 pts. each)

   a) If A is a square matrix, and $A^2 = A$, then A must be the identity matrix.
   Solution: This is false. The problem says nothing about A being invertible. The simplest counterexample is the zero matrix. If $A = 0$, then clearly, $A^2 = A = 0$. There are other counterexamples which are not identically zero, like

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

   b) If the second column of B is $[0\ 0\ 1\ 0]^{\mathrm{T}}$, then the second column of the product AB is equal to the third column of A.
   Solution: This is true. It is the column rule. Suppose the 4 columns of A are denoted $[v_1\, v_2\, v_3\, v_4]$. The second column of the product AB is the product of A and the second column of B. Since the second column of B is

$$w = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

the column rule for matrix multiplication gives that Aw will be $0v_1+0v_2+1v_3+0v_4 = v_3$, the third column of A.

c) The product of two symmetric matrices is symmetric.

Solution: This is false. Simple counterexamples can be found by choosing 2×2 symmetric matrices whose product does not commute under matrix multiplication. In general, if A and B are symmetric, then we have

$$\begin{aligned} (AB)^{\mathrm{T}} &= B^{\mathrm{T}}A^{\mathrm{T}} \\ &= BA \\ &\neq AB \end{aligned}$$

Thus, the product of two symmetric matrices A and B is symmetric if and only if $AB = BA$.