Linear Algebra
Midterm: October 22, 2001

Remember to show your work.

This exam is closed-book, closed notes, no calculators. If you find that the arithmetic for a given problem seems complicated, go back and check your work: I have written the exam so that the calculations are not hard.

Wherever possible, I recommend that you check your arithmetic by plugging in to the original equations.

1. (a) $A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 6 \\ 1 & -2 & -5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ -9 \\ 9 \end{bmatrix}$. Is $\mathbf{b} \in \text{Col } A$? If so, show how to write $\mathbf{b}$ as a linear combination of the columns of A.

(b) Does the equation $A\mathbf{x} = \mathbf{b}$ have a unique solution $\forall\ \mathbf{b} \in \mathbf{R}^3$? Why?

(c) Is A^T invertible? Why?

2. (a) Describe the solution set of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where $A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 0 & -2 & -6 \end{bmatrix}$.

(b) Give a geometric description of the solution set you found in (a).

(c) Describe the solution set of $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ -10 \end{bmatrix}$ in parametric vector form.

(d)Give a geometric description of the solution set you found in (c). How is your answer different from your answer to part (b) of this question?

3. (a) A matrix is symmetric if it is equal to its transpose: $A^T = A$. Given any square matrix B, show that $A = B + B^T$ is symmetric.

(b) A 3×3 matrix has 9 entries. If the matrix is symmetric, how many of these entries can be chosen independently? Why?

(c) Generalize your answer to part (b). How many entries in an $n \times n$ symmetric matrix can be chosen independently? Why?

(d) A matrix is skew-symmetric if it is equal to the *negative* of its transpose: $K^T = -K$. Given any square matrix B, show that $K = B - B^T$ is skew-symmetric.

4. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ 5 & 2 & 0 \end{bmatrix}$. (a) Describe the elementary row operations which transform A into I.

(b) Find A^{-1}.

(c) Suppose $BC = A$. Find C^{-1}.

5. Miscellaneaous Proofs.

(a) The matrices A, B, and C are $n \times n$. Prove that $(AB)C = A(BC)$.

(b) Prove that a given vector $\mathbf{x} \in \mathbf{R}^n$ satisfies $\mathbf{x}^T\mathbf{y} = 0$ for all $\mathbf{y} \in \mathbf{R}^n$ if and only if $\mathbf{x}$ is the zero vector.