Linear Algebra

Midterm Solutions

1. (a) $A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 6 \\ 1 & -2 & -5 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 11 \\ -9 \\ 9 \end{bmatrix}$. Is $\mathbf{b} \in \text{Col } A$? If so, show how to write $\mathbf{b}$ as a linear combination of the columns of A

Row reduce to find $[A \ \mathbf{b}] \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. So yes, $\mathbf{b} \in \text{Col } A$. Weights $x_1 = 1$, $x_2 = 1$, $x_3 = -2$ give $\mathbf{b}$ as a linear combination of the columns of A.

(b) Does the equation $A\mathbf{x} = \mathbf{b}$ have a unique solution $\forall \ \mathbf{b} \in \mathbf{R}^3$? Why?

Yes, it does have a unique solution $\forall \ \mathbf{b} \in \mathbf{R}^3$, because the echelon form of A has one pivot in each row, so the system is always consistent. This means A is invertible, since $A \sim I$, and Col $A = \mathbf{R}^3$.

(c) Is A^T invertible? Why?

Yes, since A is invertible (see part (b)) then A^T is also invertible.
Some of you row-reduced A^T. This approach is valid, but takes much longer than using the fact that A is invertible.

2. (a) Describe the solution set of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where $A = \begin{bmatrix} 1 & 2 & 4 \\ -2 & -3 & -5 \\ 0 & -2 & -6 \end{bmatrix}$.

Row reduce to find $[A \ \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Thus x_3 is a free variable, and the parametric solution is $\mathbf{x} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = x_3\mathbf{v}$. Some of you didn't fully row reduce, and wrote a parametric solution with 2 vectors instead of only 1. This is a serious error.

(b) Give a geometric description of the solution set you found in (a).

The solution set is a line through the origin and the vector $\mathbf{v}$.

(c) Describe the solution set of $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ -10 \end{bmatrix}$ in parametric vector form.

Row reduce as above to find $[A \ \mathbf{b}] \sim \begin{bmatrix} 1 & 0 & -2 & -9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Thus x_3 is a free variable, and the parametric solution is $\mathbf{x} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + \begin{bmatrix} -9 \\ 5 \\ 0 \end{bmatrix} = x_3\mathbf{v} + \mathbf{p}$.

(d)Give a geometric description of the solution set you found in (c). How is your answer different from your answer to part (b) of this question?

The solution set is a line through $\mathbf{p}$ parallel to the line through the origin and $\mathbf{v}$. It is the homogenous solution set of part (b) plus the particular solution $\mathbf{p}$.

3. (a) A matrix is symmetric if it is equal to its transpose: $A^T = A$. Given any square matrix B, show that $A = B + B^T$ is symmetric.

If A is symmetric then $A^T = A$. For any B, $A^T = (B + B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T = A$. Therefore A is symmetric no matter what B is.

(b) A 3×3 matrix has 9 entries. If the matrix is symmetric, how many of these entries can be chosen independently? Why?

There are 6 free entries. If the matrix is symmetric, $A^T = A$. Therefore $a_{ij} = a_{ji}$, which means that entries above the diagonal and entries below the diagonal are linked:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \; A^T = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

$A^T = A$ means that $b = d$, $c = g$, and $f = i$. What's on the diagonal doesn't matter. Therefore instead of having 9 entries which can be chosen independently, 3 entries are fixed based on the other 6.

(c) Generalize your answer to part (b). How many entries in an $n \times n$ symmetric matrix can be chosen independently? Why?

There are $(n^2+n)/2$ independent (free) entries. The n diagonal entries are free. Off the diagonal, there are $n^2 - n$ entries. Half of these are fixed by the other half. So there are $(n^2 - n)/2$ entries which are fixed based on the other $(n^2 - n)/2 + n = (n^2 + n)/2$.

(d) A matrix is skew-symmetric if it is equal to the *negative* of its transpose: $K^T = -K$. Given any square matrix B, show that $K = B - B^T$ is skew-symmetric.

If K is skew=symmetric then $K^T = -K$. For any B, $K^T = (B - B^T)^T = B^T - (B^T)^T = B^T - B = -(B - B^T) = -K$. Therefore K is skew=symmetric no matter what B is.

4. $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & 1 \\ 5 & 2 & 0 \end{bmatrix}$. (a) Describe the elementary row operations which transform A into I.

Interchange the 1st row with the 2nd row, then interchange the new 2nd row with the 3d row. There are other ways to do this.

(b) Find A^{-1}.

Perform the row operations on I. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

(c) Suppose $BC = A$. Find C^{-1}.

Use $(BC)^{-1} = C^{-1}B^{-1} = A^{-1}$. Multiply both sides *on the right* by B to find $C^{-1} = A^{-1}B =$

$\begin{bmatrix} 2 & 0 & 1 \\ 5 & 2 & 0 \\ 1 & 3 & -1 \end{bmatrix}$. Note that this is just the same row interchanges performed on B that you performed on A in part (a). No credit for multiplying the matrices in the wrong order.

5. Miscellaneaous Proofs.
(a) The matrices A, B, and C are $n \times n$. Prove that $(AB)C = A(BC)$.

The i, j entry of $(AB)C$ is:

$$\begin{aligned} ((AB)C)_{ij} &= \sum_{l=1}^{n} (AB)_{il} c_{lj} \\ &= \sum_{l=1}^{n} \left(\sum_{k=1}^{n} a_{ik} b_{kl} \right) c_{lj} \\ &= \sum_{l=1}^{n} \sum_{k=1}^{n} a_{ik} b_{kl} c_{lj} \\ &= \sum_{k=1}^{n} a_{ik} \sum_{l=1}^{n} b_{kl} c_{lj} \\ &= \sum_{k=1}^{n} a_{ik} (BC)_{kj} \\ &= (A(BC))_{ij} \end{aligned}$$

This holds for every component, therefore $(AB)C = A(BC)$.

(b) Prove that a given vector $\mathbf{x} \in \mathbf{R}^n$ satisfies $\mathbf{x}^T\mathbf{y} = 0$ for all $\mathbf{y} \in \mathbf{R}^n$ if and only if $\mathbf{x}$ is the zero vector.

The inner product

$$\begin{aligned} \mathbf{x}^T\mathbf{y} &= \sum_{i=1}^{n} x_i y_i \\ &= x_1 y_1 + x_2 y_2 + \cdots + x_n y_n \end{aligned}$$

(i) If $\mathbf{x} = \mathbf{0}$, then each of these terms is zero, so $\mathbf{x}^T\mathbf{y} = 0$. (ii) Now do the converse. If $\mathbf{x}^T\mathbf{y} = 0 \ \forall \ \mathbf{y}$, then in general each term in the sum will have $y_i \neq 0$. For the sum to be zero, each term must be zero, and therefore x_i must be zero for each i. Therefore $\mathbf{x}$ must be the zero vector.