Exam I Solutions
APPM 3310
Spring 2002

1. To find a basis for the subspace of $\mathbb{R}^4$ that contains all vectors such that $x_1 = 2x_3$ and $x_2 = -x_4$, write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ -x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$

the vectors that appeared on the right are a perfect candidate for a basis, and the above already shows that they span the subspace. To show they are linearly independent, try to find a nontrivial solution to

$$c_1 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 \\ -c_2 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

But this equation is only satisifed if $c_1 = c_2 = 0$, hence the vectors are linearly independent, and thus are a basis for our subspace.

2. Let $C = AB$, where A and B are symmetric, $n \times n$ matrices. Then,

$$C^T = (AB)^T = B^T A^T = BA\,,$$

since A and B are symmetric. Now, C is symmetric if and only if $C = C^T$, which is true if and only if $AB = BA$, which is not true in general because matrix multiplication is not commutative, in general. To convince yourself that a product of symmetric matrices is not generally symmetric, consider the following example:

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 7 & 14 \end{bmatrix}.$$

The product is not symmetric.

3. To find the value of c that makes the following system solveable:

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & -1 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ c \end{bmatrix},$$

use elimination to transform A to U, and the system becomes

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & c-7 \end{array}\right].$$

To make the system solveable, we must impose the condition $c-7=0$. Otherwise, if you made an arithmetic mistake, so that the last row of U was *not* all zero, then there is no condition on c. That is, if U has a full set of pivots, then the system can be solved for *any* value of c, which would have been the correct answer if you did not get all zeroes in the last row of U.

4. The inverse of the matrix

$$A = \begin{bmatrix} 1/2 & 0 & \sqrt{3}/2 \\ 0 & 1 & 0 \\ \sqrt{3}/2 & 0 & -1/2 \end{bmatrix},$$

turns out to be itself, that is $A^{-1} = A$. Since this is the case, $A^2 = I$, and so $A^3 = A \cdot A \cdot A = A \cdot A^2 = AI = A$.

5. Concerning the following true/false questions:

   (a) If A has n columns and m rows then the nullspace of A is a subspace of $\mathbb{R}^m$, is **False**. The nullspace is the set of all x such that $Ax = 0$. Since x must have n rows and 1 column, it follows that the nullspace is a subspace of $\mathbb{R}^n$.

   (b) Matrix multiplication cannot be carried out with two $m \times n$ matrices if $m \neq n$, is **True**. To multiply two matrices, the number of columns in the first matrix must equal the number of rows in the second.

   (c) The zero vector is an element of every subspace of a vector space, is **True**. A subspace of a vector space is a subset of a vector space, which is also a vector space in its own right. One of the requirements of a vector space is that it contains a zero vector, hence a subspace must also contain the zero vector. Another way to see this is by closure. Since for any x in the subspace, $c \odot x$ must also be in the subspace, taking $c = 0$ implies the zero vector must be in the subspace.

(d) Let A be an $n \times n$ matrix. If $Ax_1 = Ax_2$, with $x_1 \neq x_2$, then A is not invertible, is **True**. The whole idea of "invertibility" is that you can get back to where you came from. That is, if $Ax = b$, I can start from x and go to b (multiplication by A does that), or under the right conditions I can also start from b and go back to x (multiplication by A^{-1} does that, if it exists.) What can go wrong is that two different vectors, x_1 and x_2, both go to the *same* b, in which case we can't decide which one of them to go back to, and this is precisely when A^{-1} fails to exist.