Exam II Solutions

APPM 3310

Spring 2002

1. Suppose $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right\}$ is a basis for a subspace V of $\mathbb{R}^4$, and let $V^\perp$ denote the orthogonal complement of V.

   (a) (5pts) To find the dimension of $V^\perp$, it is helpful to recall the relationship $\dim(V) + \dim(V^\perp) = \dim(\mathbb{R}^4)$. We know that the dimension of a vector space/subspace is equal to the number of vectors in a basis for it, which is how we know $\dim(V) = 2$ and $\dim(\mathbb{R}^4) = 4$, and hence $\dim(V^\perp) = 2$.

   (b) (20 pts) To find a basis for $V^\perp$, recall its definition. It consists of all vectors orthogonal to V, that is, all vectors orthogonal to each vector in a basis for V. Thus, $x \in V^\perp$ if and only if

   $$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

   After one step of elimination, we obtain the upper triangular

   $$U = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 2 \end{bmatrix}.$$

   Pivots are 1 and -2, in 1st and 3rd columns, respectively, hence our basic variables are x_1 and x_3, and we solve for these in terms of x_2 and x_4 to obtain

   $$x \in V^\perp \quad \Longleftrightarrow \quad x = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

   The vectors that appeared are a basis for $V^\perp$. (In other words, first set $x_2 = 1$ and $x_4 = 0$, then vice versa, to obtain two vectors which are a basis.)

(c) (20pts) Let the basis vectors be the columns of a matrix B, then the projection matrix P equals $B(B^TB)^{-1}B^T$. Thus,

$$B^TB = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \qquad (B^TB)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix},$$

$$\begin{aligned} P &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/3 & 0 & -1/3 & -1/3 \\ 0 & 1 & 0 & 0 \\ -1/3 & 0 & 1/3 & 1/3 \\ -1/3 & 0 & 1/3 & 1/3 \end{bmatrix}. \end{aligned}$$

2. Assume P is a projection matrix.

(a) (5pts) Two properties satisfied by every projection matrix are:

i. $P^2 = P$ (P is idempotent),

ii. $P^T = P$ (P is symmetric).

(I ended up charging only 5 pts for both answers.)

(b) (20pts) To show $P_1 = I - P$ is also a projection matrix, it is sufficient to show it satisfies both of the above properties.

i. $P_1^2 = (I-P)(I-P) = I-2P+P^2 = I-2P+P = I-P = P_1$,

ii. $P_1^T = (I-P)^T = I^T - P^T = I - P = P_1$.

3. (20pts) Given that $\{\alpha_1, \alpha_2\}$ is a basis for a subspace W of $\mathbb{R}^4$, where

$$\alpha_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \qquad \alpha_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$

we construct an orthonormal basis for W by using the Gram-Schmidt process to construct mutually orthogonal β_1 and β_2, then normalize to obtain q_1 and q_2, which are an orthonormal basis for W. Thus,

$$\begin{aligned}
\beta_1 &= \alpha_1 \\
\beta_2 &= \alpha_2 - \left(\frac{\beta_1^T \alpha_2}{\beta_1^T \beta_1}\right) \beta_1 \\
&= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad = \quad \begin{bmatrix} 2/3 \\ -1 \\ -2/3 \\ 2/3 \end{bmatrix}.
\end{aligned}$$

Normalizing these, we have

$$q_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \qquad q_2 = \frac{1}{\sqrt{21}} \begin{bmatrix} 2 \\ -3 \\ -2 \\ 2 \end{bmatrix},$$

4. (10pts) For $x, y \in \mathbb{R}^n$, a simple calculation proves the identity

$$\frac{1}{2}\left(\|x+y\|^2 + \|x-y\|^2\right) = \|x\|^2 + \|y\|^2 .$$

Expanding the norm as inner product, we have

$$\begin{aligned}
&\frac{1}{2}\left(\|x+y\|^2 + \|x-y\|^2\right) \\
&= \frac{1}{2}\left[(x+y)^T(x+y) + (x-y)^T(x-y)\right] \\
&= \frac{1}{2}\left(x^T x + 2x^T y + y^T y + x^T x - 2x^T y + y^T y\right) \\
&= x^T x + y^T y \quad = \quad \|x\|^2 + \|y\|^2 .
\end{aligned}$$