

APPM 3310, Spring 2002

Study Guide for Exam III

This exam will focus exclusively on material from chapters 4 and 5 of our text that was covered in class. Specifically, see homeworks 8, 9, and 10. As always, you still need to know how to execute Gauss/Jordan elimination, you still need to be familiar with the notion of subspace, dimension of a subspace, basis, etc. The material is too interconnected for anybody to be able to ignore the first part of the course.

Determinants

Computation of Determinants. You should be able to compute determinants either using row operations (elimination), or cofactor expansions.

1. Row Operations:
 - (a) Adding a multiple of one row to another row of a matrix does not alter its determinant.
 - (b) Exchanging two rows of a matrix multiplies its determinant by -1 .
 - (c) Multiplying a row of a matrix by a scalar multiplies its determinant by the scalar.
2. Cofactor Expansions: be sure to keep an eye on the sign $(-1)^{i+j}$ that goes in front of the determinant of the matrix with i th row and j th column deleted.

Some basic theorems. Here are a few facts concerning determinants that we should all be familiar with.

1. The determinant of an upper/lower triangular matrix (this includes diagonal matrices) is the product of the elements on the main diagonal. This is, of course, what allows us to evaluate the determinant using the elimination method, since we first transform the matrix to upper triangular form, then multiply down the diagonal.
2. The determinant of the transpose of a matrix is equal to the determinant of the matrix: $\det(A^T) = \det(A)$.

3. The determinant of the inverse of a matrix (provided it is invertible) is the inverse of the determinant: $\det(A^{-1}) = 1/\det(A)$.
4. The determinant of a product of matrices is the product of the determinants: $\det(AB) = \det(A)\det(B)$.
5. A matrix with determinant equal to zero is singular (not invertible), and vice versa. Another way of saying the same thing: a matrix is nonsingular (invertible) if and only if its determinant is nonzero.
6. If the size of A is $n \times n$, then $\det(tA) = t^n \det(A)$.

Eigenvalues and Eigenvectors

1. The whole concept is very simple: given a matrix A , there are certain directions which are *invariant* under multiplication by A . The length of the vector may change, but not the direction. Written in mathematical notation this is just the basic equation $Ax = \lambda x$. In matrix form, we have $AS = S\Lambda$, where the columns of S are eigenvectors and the diagonal entries of diagonal matrix Λ are eigenvalues. Of course, an eigenvector must be nonzero, since the “direction” of a zero vector is meaningless.
2. We find eigenvalues, λ , by solving

$$p(\lambda) = \det(\lambda I - A) = 0. \tag{1}$$

That is, we find the zeroes of the characteristic polynomial $p(\lambda)$, which is defined above. We define it in just this way, with the minus sign on A , so that the leading term of the polynomial will have coefficient 1. That is,

$$p(\lambda) = \lambda^n + (\text{lower order terms}),$$

which is more natural (at least to me).

3. We find the eigenvectors, x , by solving

$$(A - \lambda I)x = 0. \tag{2}$$

Notice that I have now multiplied equation (2) by -1 , relative to the previous equation (1), so that the minus sign is no longer on A . However, this is perfectly okay since the other side of the equation is zero,

and it is now in a form that works better when we get to the generalized eigenvectors later on. This should cause no confusion!

4. As should be apparent to all upon examination of (2), the eigenvectors of A belonging to λ form a subspace, which is none other than the nullspace of $(A - \lambda I)$. We sometimes refer to this nullspace as the eigenspace of A belonging to λ . The eigenvectors we will use are a basis for the eigenspace. The number of linearly independent eigenvectors belonging to a given λ is equal to the dimension of the eigenspace, which is equal to the number of vectors in a basis.
5. Collect all the *linearly independent* eigenvectors we found, i.e. the basis for the eigenspace for each eigenvalue, and place them into the columns of a matrix called S . Collect all the eigenvalues they belong to and put them into the diagonal entries of a diagonal matrix called Λ . Be sure to put the eigenvectors and the eigenvalues to which they belong in the corresponding columns. Then you have the matrix form of the basic equation,

$$AS = S\Lambda. \tag{3}$$

6. Note that equation (3) is always satisfied. *If* the matrix S is invertible, *then* we can multiply on both sides of (3) by the inverse, to obtain

$$A = S\Lambda S^{-1}, \quad \text{or } \Lambda = S^{-1}AS. \tag{4}$$

In this case, we say that A is *diagonalizable*, it is *similar* to a diagonal matrix.

7. If S is not invertible, then A is not diagonalizable. Since we assumed the columns of S are linearly independent, this can only happen if S doesn't have enough columns. It isn't a square matrix because there are not enough linearly independent eigenvectors. This can only happen if there are repeated eigenvalues, i.e. having a multiplicity greater than one. In this case, we can find more columns to augment those in S , to obtain a square matrix M , which is invertible. In place of (3), we will have

$$AM = MJ,$$

where J is bidiagonal. J has eigenvalues along its main diagonal, the same as Λ , but it has 1s, as needed, on its first super-diagonal.

8. The form of J tells the equation(s) that must be solved to find the additional column(s). Suppose x is an eigenvector for λ , and that the multiplicity of λ as a root of $p(\lambda)$ is 2, but that the dimension of the eigenspace for λ is 1, so that the single vector x is a basis. As it happens in this case, though we didn't prove it, x is also in the column space of $(A - \lambda I)$, which means that we can find a vector y such that

$$(A - \lambda I)y = x. \quad (5)$$

Vector y is called a *generalized eigenvector*. Rearranging the above equation, we have

$$Ay = \lambda y + x,$$

and $Ax = \lambda x$ since x is an eigenvector. If x and y are two consecutive columns of M , then the equation $AM = MJ$ will look like

$$AM = A[\cdots | x | y | \cdots] = [\cdots | \lambda x | x + \lambda y | \cdots] = M \begin{bmatrix} \cdots & & & 0 \\ & \lambda & 1 & \\ & & \lambda & \\ 0 & & & \cdots \end{bmatrix}$$

The matrix on the extreme left is J (well, a piece of it is displayed, anyway). Compare the form of J with equation (5), and you should be able to see that they are equivalent.

9. The matrix J is called the Jordan form, and we will accept without proof that every matrix not diagonalizable is nevertheless similar to a bidiagonal matrix. (If A is diagonalizable, then we say that $J = \Lambda$, but that's not very important for us.)
10. One last thought about generalized eigenvectors: we can always find as many as we need for a given eigenvalue, up to the multiplicity of the eigenvalue. We simply solve the following sequence of problems:

$$\begin{aligned} (A - \lambda I)x_1 &= 0, & (x_1 \text{ is the eigenvector}) \\ (A - \lambda I)x_2 &= x_1, \\ (A - \lambda I)x_3 &= x_2, \end{aligned}$$

and so on.

No calculators, as you very well know by now.