Midterm 2 Solutions

1. Consider $A = \begin{bmatrix} 1 & 5 & 0 & -3 \\ 2 & 10 & 1 & -4 \\ -1 & -5 & 1 & 5 \end{bmatrix}$. Use this definition of A for all parts of this problem.

Give a basis for (a) the row space of A, (b) the column space of A, (c) the null space of A, (d) the left null space of A.

Row reduction gives $A \sim U = \begin{bmatrix} 1 & 5 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A basis for the row space is the first two nonzero rows of U (or A), that is, $\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$. For the column space, use the pivot columns (the first and third columns) of A: $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$. (Note: zero points for using the first and third columns of U.) For the null space, the columns 2 and 4 have free variables, leading to a basis set $\left\{ \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$. To find the left null space, row reduce $A^T \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The third column corresponds to a free variable, so the basis vector is $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(e) What is the rank of A? (f) What is the rank of A^T?

A has two pivots, so its rank $= 2$. Similarly, A^T has two pivots, so its rank is 2. The purpose of this question was to help you find errors: you should remember that rank $A =$ rank A^T always! So if you made an arithmetic error and claimed the ranks were different, you did not receive full credit.

2. You are given 4 vectors, $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \mathbf{R}^4$. This set spans a d-dimensional subspace of $\mathbf{R}^4$.

(a) When, if ever, is $d < 1$?

When all 4 vectors are the zero vector; when the matrix $A = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4]$ is the zero matrix.

(b) When, if ever, is $d = 1$?

When all 4 vectors point along the same line; when all 4 vectors are multiples of each other; when the set has only one linearly independent vector; when the matrix $A = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4]$ is rank 1 (one pivot).

(c) When, if ever, is $d = 2$?

When all 4 vectors lie in a same plane; when the set has only two linearly independent vectors; when the matrix $A = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4]$ is rank 2 (two pivots).

(d) When, if ever, is $d = 3$?

When the set has only three linearly independent vectors; when the matrix $A = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4]$ is rank 3 (three pivots).

(e) When, if ever, is $d = 4$?

When the vectors are linearly independent; when the matrix $A = [\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3\ \mathbf{x}_4]$ is full rank (four pivots).

(f) When, if ever, is $d > 4$?

Never: it is impossible for vectors $\in \mathbf{R}^4$ to span a space with dimension > 4.

(g) In which case is the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ a basis, and for what space?
In case (e), when the vectors are linearly independent, they span $\mathbf{R}^4$ and are thus a basis for $\mathbf{R}^4$.

3. Describe the linear transformations of $\mathbf{R}^2$ performed by the following matrices. (By "describe" I mean explain how the space is geometrically transformed. A sketch is recommended.)
(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$
The transformation A reflects about the x axis. The transformation B rotates clockwise by 90 degrees. The transformation C *shears* in the x-direction. (Note: many people had difficulty recognizing the shear transformation, and wrote things like the transformation "rotates and stretches the y-axis" or "stretches the x-axis." I was lenient in grading wording like this if the sketch was correct, but make sure you understand what shear means for the final.)
(d) Describe the composite linear transformations performed by AB and by BA. Are they the same? Why or why not?
First, $AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. This transformation first rotates clockwise by 90 degrees, then reflects through the x-axis. The result, AB, is a reflection through the line $y = x$. But $BA = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. When the order is reversed (first reflect, then rotate), the result is a reflection through the origin—you switch the x and y coordinates, then make both negative. Clearly $AB \neq BA$, which demonstrates that reflections and rotations do not commute with each other. (Note: zero points if you got the order of the transformations wrong in the composition.)

4. Problems from the previous midterm, revisited.
(a) Prove that if a matrix is symmetric, its column space is orthogonal to its null space.
First proof: the fundamental theorem of linear algebra tells us that the row space of a matrix is orthogonal to its null space. For a symmetric matrix, the row space and column space are the same. Therefore the column space is also orthogonal to the null space.
Second proof: A vector $\mathbf{x}$ in the null space satisfies $A\mathbf{x} = \mathbf{0}$, while a vector $\mathbf{b}$ in the column space can be written $A\mathbf{y} = \mathbf{b}$. Take the inner product

$$\begin{aligned} \mathbf{b}^T\mathbf{x} &= (A\mathbf{y})^T\mathbf{x}, \\ &= \mathbf{y}^T A^T \mathbf{x}, \\ &= \mathbf{y}^T A\mathbf{x}, \\ &= \mathbf{y}^T \mathbf{0}, \\ &= 0. \end{aligned}$$

Where we have used the fact that $A^T = A$ for a symmetric matrix and that $A\mathbf{x} = \mathbf{0}$.

(b) If you know the determinant of an $n \times n$ matrix A, what is $\det(-A)$?
First (more elegant) proof: We can write

$$\begin{aligned} |-A| &= |-IA|, \\ &= |-I||A|. \end{aligned}$$

But $|-I| = (-1)^n$, because the determinant of a diagonal matrix is the product of the diagonal elements, and $-I$ has n diagonal elements, each $= -1$. Therefore $|-A| = (-1)^n|A|$.

Second proof: We know the determinant is linear in each row. Therefore if we factor out -1 from the first row, the determinant changes sign. Then we factor out -1 from every other row, changing the sign each time. Ultimately we have factored out -1 n times, so $|-A| = (-1)^n|A|$.

(c) Recall that if a matrix K is **skew-symmetric** then $K^T = -K$. Use your answer to part (b) to prove that all 3×3 skew-symmetric matrices have the same determinant, and find the value of the determinant.

From above, we have det $-K = -$det K. However, we also know that in general det $K^T =$ det K, which together imply det $K = -$ det K. The only way the determinant can equal the negative of itself is if it is zero. Therefore det $K = 0$ for a 3×3 skew-symmetric matrix (and in fact for any $n \times n$ skew-symmetric matrix when n is odd.