Matrix Methods

Spring 2003 Final Exam with Solutions

1. Consider the system $B\mathbf{x} = \mathbf{b}$, where $B = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}$. (a) Find the set of all possible least-squares solutions of this system.

(5 points) Solve $B^T B\hat{\mathbf{x}} = B^T\mathbf{b}$, and $BB^T = \begin{bmatrix} 3 & 0 \\ 0 & 24 \end{bmatrix}$ $B^T\mathbf{b} = \begin{bmatrix} 9 \\ 12 \end{bmatrix}$

$\begin{bmatrix} 3 & 0 & 9 \\ 0 & 24 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \end{bmatrix}$

so $\hat{x}_1 = 3$, $\hat{x}_2 = 1/2$

The solution is $\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ 1/2 \end{bmatrix}$. Two points off if you rescale the solution, four points off if you think $\hat{\mathbf{b}}$ is the solution.

(b) Find the projection of $\mathbf{b}$ onto the column space of B.

(3 points) The quickest way to solve is to note that $\hat{\mathbf{b}} = B\hat{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 4 \end{bmatrix}$ is the projection of $\mathbf{b}$ onto the column space of B. You could also note that the columns of B are orthogonal and compute $\hat{\mathbf{b}}$ by orthogonal projection. (2 points off if you only project onto one column of B).

(c) Find the least-squares error.

(3 points) The error is $||\mathbf{b} - \hat{\mathbf{b}}||$. The vector $\mathbf{b} - \hat{\mathbf{b}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, so $||\mathbf{b} - \hat{\mathbf{b}}|| = \sqrt{2}$. Two points off if you rescaled a vector or didn't find the length of the vector.

2. You are given a matrix A and three linearly independent vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. The matrix A has only two distinct eigenvalues, the eigenspace of λ_1 is Span$\{\mathbf{u}, \mathbf{v}\}$, and the eigenspace of λ_2 is Span$\{\mathbf{w}\}$.

(a) Is it possible that A is a 2×2 matrix? If so, is it diagonalizable? (5 points)

A cannot be a 2×2 matrix because it is impossible to have 3 linearly independent vectors in $\mathbf{R^2}$.

(b) Is it possible that A is a 3×3 matrix? If so, is it diagonalizable?

A could be a 3×3 matrix, because 3 linearly indpoendent bectors in $\mathbf{R^3}$ are possible. To diagonalize A we would need 3 linearly independent eigenvectors, which we have. So A is diagonalizable.

(c) Is it possible that A is a 4×4 matrix? If so, is it diagonalizable?

A could be 4×4, but it wouldn't be diagonalizable because it doesn't have 4 linearly independent eigenvectors.

3. Consider $A = \frac{1}{4}\begin{bmatrix} -3 & 5 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} -3/4 & 5/4 \\ 5/4 & -3/4 \end{bmatrix}$. Use this definition of A for all parts of this problem.

(a) What is A^T? (2 points)

$A^T = \begin{bmatrix} -3/4 & 5/4 \\ 5/4 & -3/4 \end{bmatrix} = A$. The point of this part of the problem is to recognize that A is symmetric, and thus is diagonalizable with real eigenvalues and orthogonal eigenvectors

(b) Calculate the eigenvalues of A. (3 points)

$\det(A - \lambda I) = \frac{1}{4}\begin{vmatrix} -3-\lambda & 5 \\ 5 & -3-\lambda \end{vmatrix} = \frac{1}{4}(\lambda^2 + 6\lambda - 16) = \frac{1}{4}(\lambda+8)(\lambda-2) = 0$. So the eigenvalues are $\lambda_1 = -8$ and $\lambda_2 = 2$.

(c) Calculate the corresponding eigenvectors. (5 points)

$(A - \lambda_1 I)\mathbf{x}_1 = \frac{1}{4}\begin{vmatrix} 5 & 5 \\ 5 & 5 \end{vmatrix}\mathbf{x}_1 = \mathbf{0}$. Therefore $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

And $(A - \lambda_2 I)\mathbf{x}_2 = \frac{1}{4}\begin{vmatrix} -5 & 5 \\ 5 & -5 \end{vmatrix}\mathbf{x}_2 = \mathbf{0}$. Therefore $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(d) Are these eigenvectors linearly independent? Why? (3 points)

Yes, they are linearly independent. Eigenvectors corresponding to distinct eigenvalues are always linearly independent.

(e) Are these eigenvectors orthogonal? If not, make them orthogonal. (3 points)

Yes, they are orthogonal, which we can check by noting that $\mathbf{x}_1^T\mathbf{x}_2 = 0$. They must be orthogonal, since the spectral theorem says that for a real symmetric matrix, eigenvectors corresponding to distinct eigenvectors are orthogonal.

(f) Are these eigenvectors orthonormal? If not, normalize them. (3 points)

In the form given above, they are not normalized. The normalized versions are $\mathbf{x}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(g) Are these eigenvectors a basis for some vector space? If not, why not? If so, what vector space? (3 points)

These eigenvectors are a basis for $\mathbf{R}^2$, since they are linearly indpendent and span the space.

(h) Consider the dynamical system $\mathbf{x}_k = A\mathbf{x}_{k-1}$. What is the limiting behavior of $\mathbf{x}_k$ as $k \to \infty$ for $\mathbf{x}_0 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$? (3 points)

The initial condition $\mathbf{x}_0$ is the eigenvector for $\lambda_2 = 2$. Therefore the solution for all k is $\mathbf{x}_k = 2^k\mathbf{x}_0 = 2^k\begin{bmatrix} 5 \\ 5 \end{bmatrix}$. This solution $\to \infty$ for large k (unstable).

(i) Consider the dynamical system $\mathbf{x}_k = A\mathbf{x}_{k-1}$. What is the limiting behavior of $\mathbf{x}_k$ as $k \to \infty$ for $\mathbf{x}_0 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$? (3 points)

The initial condition $\mathbf{x}_0$ is the eigenvector for $\lambda_1 = -8$. Therefore the solution for all k is

$\mathbf{x}_k = (-8)^k \mathbf{x}_0 = (-8)^k \begin{bmatrix} -3 \\ 3 \end{bmatrix}$. This solution $\to \infty$ for large k (unstable), although it changes direction every step.

(j) Consider the dynamical system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$. What is the limiting behavior of $\mathbf{x}(t)$ as $t \to \infty$ for $\mathbf{x}_0 = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$? (3 points)

As in part (h), the initial condition $\mathbf{x}_0$ is the eigenvector for $\lambda_2 = 2$. Therefore the solution for all time is $\mathbf{x}(t) = e^{2t}\mathbf{x}_0 = e^{2t} \begin{bmatrix} 5 \\ 5 \end{bmatrix}$. This solution $\to \infty$ for large t (unstable).

(k) Consider the dynamical system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$. What is the limiting behavior of $\mathbf{x}(t)$ as $t \to \infty$ for $\mathbf{x}_0 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$? (3 points)

As in part (i), the initial condition $\mathbf{x}_0$ is the eigenvector for $\lambda_1 = -8$. Therefore the solution for all time is $\mathbf{x}(t) = e^{-8t}\mathbf{x}_0 = e^{-8t} \begin{bmatrix} -3 \\ 3 \end{bmatrix}$. This solution $\to 0$ for large t (stable).

(l) Find matrices S, Λ, and S^{-1} such that $A = S\Lambda S^{-1}$. (3 points)

Use the eigenvalues and eigenvectors from above. The matrix $S = Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ with $S^{-1} = Q^T$, and $\Lambda = \begin{bmatrix} -8 & 0 \\ 0 & 2 \end{bmatrix}$.

4. In this problem you will work with the space of 2×2 matrices $\mathbf{M}_{2\times 2}$. You may take as given that $\mathbf{M}_{2\times 2}$ is a vector space, under the usual operations of addition of matrices and multiplication by real scalars. (a) Is the set of all rank-one 2×2 matrices a subspace of $\mathbf{M}_{2\times 2}$? Why or why not? (5 points)

It is not a subspace because it doesn't include the zero matrix. (The zero matrix is not a rank one matrix.)

(b) Show that the set of all upper triangular 2×2 matrices is a subspace of $\mathbf{M}_{2\times 2}$. Call this subspace V.

The set of all upper triangular 2×2 matrices is a subset of $\mathbf{M}_{2\times 2}$. A general upper triangular matrix has the form $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$. Thus the zero matrix is in V if we choose $a = b = c = 0$. If we add, $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix}$ the result is still upper triangular, so V is closed under addition. V is also closed under scalar multiplication, because $r \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ra & rb \\ 0 & rc \end{bmatrix}$ is in V. Therefore V is a subspace of $\mathbf{M}_{2\times 2}$.

(c) Find a basis for the subspace V.

A basis is $\{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. By adding these 3 matrices together with different weights, we could form any 2×2 upper triangular matrix.

(d) What is the dimension of the subspace V? Explain how this is related to the number of entries in a 2×2 upper triangular matrix that can be chosen independently.

The dimension of V is the number of basis vectors, which is 3. This is the same as the number of entries that can be chosen independently, because the entries one can choose are exactly the weights of the linear combination of basis vectors.

(e) Consider the transformation $T : V \to V$ which maps a 2×2 upper triangular matrix U to AU, where $A = \begin{bmatrix} -1 & 3 \\ 0 & 4 \end{bmatrix}$. Determine how your basis vectors from part (c) transform under T.

$$A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, A \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & 4 \end{bmatrix}$$

(f) Find the matrix of the transformation T relative to your basis from part (c).

We can describe an arbitrary 2×2 upper triangular matrix as a linear combination of the basis vectors: $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = a\ \mathbf{b}_1 + b\ \mathbf{b}_2 + c\ \mathbf{b}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}_V$. In other words, the matrix can be represented in the basis V by the coordinates $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$. In part (e), you found that under the transformation $\mathbf{b_1} \to -1\ \mathbf{b_1}$, or $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$. The transformed vector is the first column of the matrix. Since $\mathbf{b_2} \to -1\ \mathbf{b_2}$ and $\mathbf{b_3} \to 3\ \mathbf{b_2} + 4\mathbf{b}_3$, the other columns are $\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$. Thus the matrix of the transformation is $[T]_V = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$

6. Consider two nonzero vectors $\mathbf{u} \in \mathbf{R}^m$, $\mathbf{v} \in \mathbf{R}^n$. (a) Use these vectors to construct an $m \times n$ matrix M.

$M = \mathbf{u}\mathbf{v}^T$

(b) Find a basis for the column space of M.

Every column is a multiple of $\mathbf{u}$, therefore $\mathbf{u}$ is a basis of the column space.

(c) Find a basis for the row space of M.

Every row is a multiple of $\mathbf{v}$, therefore $\mathbf{v}$ is a basis for the row space

(d) What is the rank of M?

The rank is one, because there is one linearly independent column.

(e) Find an eigenvector of M and an expression for the eigenvalue (when $m = n$).

When $m = n$, then $\mathbf{u}$ is an eigenvector because $\mathbf{u}\mathbf{v}^{\mathbf{T}}\mathbf{u} = (\mathbf{v}^{\mathbf{T}}\mathbf{u})\mathbf{u}$, and the eigenvalue is $\mathbf{v}^{\mathbf{T}}\mathbf{u}$.

6. The singular value decomposition of the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the pseudo-inverse of A. (5 points)

The pseudo-inverse is $A^+ = Q_2 \Sigma^+ Q_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} = A^T$.

(b) Find the minimum length least-squares solution to $A\mathbf{x} = \mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$. (3 points)

The solution is $\mathbf{x}^+ = A^+\mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

(c) The minimum length least-squares solution $\mathbf{x}^+$ is in one of the four fundamental subspaces of A. Which one? Explain. (3 points)

The solution $\mathbf{x}^+$ is in the row space of A, by construction: the minimum length least-squares solution is always in the row space.

(d) The vector $A\mathbf{x}^+$ is in one of the four fundamental subspaces of A. Which one? Explain. (3 points)

When we multiply a matrix A times *any* vector, the result is *always* in the column space of A. Therefore $A\mathbf{x}^+ = AA^+\mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is in the column space of A. Note that in this case, we just recovered the original vector $\mathbf{b}$.

(e) Find the projection of $\mathbf{b}$ onto the column space of A. (Hint: the answers to part (c) and (d) should help you. 3 points)

Since $A\mathbf{x}^+ = AA^+\mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is in the column space of A, the projection onto the column space is just $\mathbf{b}$—the vector is *already* in the column space of A.

(f) Find the projection of $\mathbf{b}$ onto the left nullspace of A. (3 points)

The left nullspace component in general is $AA^+\mathbf{b} - \mathbf{b}$, which here is zero.

7. Proofs.

(a) If A and B are similar matrices, prove that $\det A = \det B$. (5 points)

If A and B are similar, then we can write $A = MBM^{-1}$ for some matrix M. Then

$$\begin{aligned} \det A &= \det MBM^{-1}, \\ &= \det M \det B \det M^{-1}, \end{aligned}$$

$$
\begin{aligned}
&= \frac{\det M \, \det B}{\det M}, \\
&= \det B.
\end{aligned}
$$

(b) If A is a real, symmetric matrix, prove that eigenvectors corresponding to distinct eigenvalues are orthogonal (without recourse to the spectral theorem). (10 points)

We have two eigenvectors of distinct eigenvalues: $A\mathbf{x}_1 = \lambda_1 \mathbf{x}_1$ and $A\mathbf{x}_2 = \lambda_2 \mathbf{x}_2$ with $\lambda_1 \neq \lambda_2$. Take the inner product of $\mathbf{x}_1$ with $A\mathbf{x}_2$:

$$
\begin{aligned}
\mathbf{x}_1^T A \mathbf{x}_2 &= \mathbf{x}_1^T \lambda_2 \mathbf{x}_2, \\
&= \lambda_2 \mathbf{x}_1^T \mathbf{x}_2.
\end{aligned}
$$

It is also true that

$$
\begin{aligned}
\mathbf{x}_1^T A \mathbf{x}_2 &= \mathbf{x}_1^T A^T \mathbf{x}_2, \\
&= (A\mathbf{x}_1)^T \mathbf{x}_2, \\
&= (\lambda_1 \mathbf{x}_1)^T \mathbf{x}_2, \\
&= \lambda_1 \mathbf{x}_1^T \mathbf{x}_2.
\end{aligned}
$$

This proves that $\lambda_1 \mathbf{x}_1^T \mathbf{x}_2 = \lambda_2 \mathbf{x}_1^T \mathbf{x}_2$. Now for this equation to be true, there are only two options: (a) we could have $\lambda_1 = \lambda_2$, but this is impossible, since we initially assumed that the eigenvalues are *distinct*. The only other option is (b) $\mathbf{x}_1^T \mathbf{x}_2 = 0$, implying that the eigenvectors are orthogonal.

Good luck on the rest of your finals, and have a great summer.