Matrix Methods
Midterm 1: October 1, 2003

Remember to show your work. **A correct answer without explanation receives no credit.**
This exam is closed-book, closed notes, no calculators. You are allowed a one-page cheat sheet.
If you find that the arithmetic for a given problem seems complicated, go back and check your work. I have written the exam so that the calculations are not hard.
Wherever possible, I recommend that you check your arithmetic by plugging in to the original equations.

1. Use the defined A and $\mathbf{b}$ for all parts of this problem.
   (a) Find any conditions on $\mathbf{b}$ so that $A\mathbf{x} = \mathbf{b}$ has a solution, for $A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.
   (b) Describe the nullspace of A. Give both a formula and a geometric description.
   (c) Find the general solution to $A\mathbf{x} = \mathbf{b}$, when the solution exists.
   (d) Describe the column space of A. Give both a formula and a geometric description.
   (e) What is the rank of A?
2. (a) Use elimination to find L, D, and U so that $A = LDU$ if $A = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}$.
   (b) Under what conditions on a, b, c, and d is A invertible?
3. In this problem, consider the vector space $\mathbf{M}_{2\times 2}$ of 2×2 matrices, and use the same definition of A and B in all parts of this problem.
   (a) Describe a subspace of $\mathbf{M}_{2\times 2}$ that contains $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ but not $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$.
   (b) If a subspace of $\mathbf{M}_{2\times 2}$ contains A and B, must it contain I?
4. Answer the following questions with complete explanations.
   (a) Does the set of vectors of the form $\begin{bmatrix} a \\ a+2 \\ a \end{bmatrix}$ form a vector subspace in $\mathbf{R}^3$?
   (b) If A is an $n \times n$ invertible matrix, is $(A^T)^2$ always invertible? If so, describe its inverse.
   (c) Given that A, B, and C are $n \times n$ matrices, suppose that C is invertible, $(A-B)C = 0$ and $B = A^T$. Prove that A is symmetric.

Extra Credit: Suppose A is a 2×1 matrix and B is a 1×2 matrix. Is it possible for $C = AB$ to be invertible? Explain why or why not.