APPM 3310: Matrix Methods — Final Exam — December 12, 2005

On the front of your bluebook print (1) your name, (2) your student ID number, and (3) a grading table. Show all work in your bluebook. Textbooks, class notes and calculators are not permitted, although you are allowed to use a one page reminder sheet. If you find that the arithmetic for this exam seems complicated, go back and check your work.

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1. (20 points) Here are some short answer questions. As always, explain your answer!

   (a) Suppose A is a 3×3 matrix with diagonal elements $1, 3$, and -4, and that two of its eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. What is the third eigenvalue?

   (b) If A is a square matrix and $A = 5B$, then the eigenvalues of A are five times the eigenvalues of B. True or False?

   (c) Prove that the product of the singular values of a square, nonsingular matrix A is equal to $|\det A|$.

   (d) Suppose a 3×3 matrix A has eigenvalues $\lambda = 1, 2$, and 3. What are the eigenvalues of $B = A - 2I$?

2. (30 points) Consider the system $A\mathbf{x} = \mathbf{b}$, with $A = \begin{pmatrix} 2 & 2 \\ 2 & 1 \\ -1 & -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

   (a) Find a basis for $rng(A)$.

   (b) Find a basis for $coker(A)$. Explain how your calculation is consistent with the fundamental theorem of Linear Algebra.

   (c) Does the system $A\mathbf{x} = \mathbf{b}$ have a solution? Why or why not?

   (d) Find the least squares solution to $A\mathbf{x} = \mathbf{b}$.

3. (20 Points) A few True-False questions. You **must** explain your answer!

   (a) The quadratic form $q(x, y) = x^2 + 2xy + 3y^2$ is positive definite.

   (b) The expression $\langle \mathbf{v}, \mathbf{w} \rangle = v_1 w_1 + 2v_1 w_2 + 3w_2 v_2$ is an inner product on $\mathbb{R}^2$.

   (c) If $AB = I$, then $BA = I$.

   (d) If A and B are invertible, then so is $A + B$.

   (e) If $\mathbf{v}$ and $\mathbf{w}$ are nonzero column vectors in $\mathbb{R}^n$, then $rank(\mathbf{v}\mathbf{w}^T) = 1$.

4. (20 points). Let $\mathcal{V}$ be the set of all polynomials $p(x)$ of at most cubic order with real coefficients such that $p(1) = 0$.

   (a) Show that $\mathcal{V}$ is a vector space.

   (b) Find a basis $p_i(x)$, $i = 1, \ldots$ for $\mathcal{V}$. What is the dimension of $\mathcal{V}$?

   (c) Using the L^2 inner product on the interval $[-1, 1]$, find the inner product of your basis vectors p_1 and p_2.

   (d) Use the Gram-Schmidt procedure to find a new basis vector $q_2(x)$ that is orthogonal to $p_1(x)$.

5. (30 points) For this problem let $A = \begin{pmatrix} 1 & -2 \\ 3 & -6 \end{pmatrix}$.

   (a) Find the LU decomposition of A.

   (b) Find the eigenvalues and eigenvectors of A.

   (c) "Complete matrices" can be diagonalized. Explain what this statement means. Is A complete?

   (d) Compute A^5 (Hint: you should use the results you obtained above; no credit if you just multiply A by itself five times!).

6. (30 Points). Using the same matrix A as the previous problem

   (a) Find the singular values of A.

   (b) What is the condition number of A?

   (c) Find the singular value decomposition of A.

   (d) Find the pseudoinverse A^+ of A.

EXTRA CREDIT (20 Points) A fundamental theorem that we did not cover in class is the Cayley-Hamilton Theorem which says that "a square matrix is a solution of its own characteristic equation."

1. Let $p(\lambda)$ be the characteristic equation for a matrix A. Write out the form of this equation in terms of the eigenvalues of A. The Cayley-Hamilton theorem states that $p(A) = 0$. Explain what this equation means.

2. Suppose that A is a diagonal matrix. Prove that $p(A) = 0$.

3. Suppose that A is a complete matrix. Use analysis like that in Problem 5 (d) to prove that $p(A) = 0$.

4. It is much harder to show that this result works for incomplete matrices as well. I think you've worked hard enough for now. Enjoy your break!