APPM 3310: Matrix Methods — Final Exam — May 9, 2006

On the front of your bluebook print (1) your name, (2) your student ID number, and (3) a grading table. Show all work in your bluebook. Textbooks, class notes and calculators are not permitted, although you are allowed to use one page of notes as a reminder sheet. If you find that the arithmetic for this exam seems complicated, go back and check your work.

Please sign your bluebook under the Honor Code to indicate that you have neither given nor received unauthorized assistance on this exam.

Please skip one of the 30 point problems, #2, 4, or 5. Write the number of the skipped problem on the front of your bluebook.

1. (50 points) Each of the following unrelated questions involves a concept from class. The concept is in bold-face. A complete answer for each part will include a definition of the bold-faced word. Then, use your definition in your answer to the question.

   (a) Are the 4 matrices given by $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, a **basis** of $M_{2\times 2}$?

   (b) What are the L^1, L^2, and L^∞ **norms** on $[-1, 1]$ for the function $f(x) = x - (1/4)$?

   (c) Find the **rank** of the $m \times n$ matrix $A = vw^T$, where v is a nonzero $m \times 1$ vector and w^T is a nonzero $1 \times n$ row vector.

   (d) Is the function $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ x - y \end{pmatrix}$ a **linear function**? Explain.

2. (30 points) For this problem, let $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$.

   (a) Find $\operatorname{corng}(A)$ and $\ker(A)$.

   (b) Find conditions on vector b so that $Ax = b$ has a solution.

   (c) For $b = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ find the general solution $Ax = b$ given by $x = w + z$ with $w \in \operatorname{corng}(A)$ and $z \in \ker(A)$.

3. (35 points) As you know, an $n \times n$ matrix is symmetric if $A = A^T$. An $n \times n$ matrix J is called skew-symmetric if $J = -J^T$. (Skew-symmetric matrices arise in many applications, particularly in physics.)

   (a) Suppose a matrix J is skew-symmetric. What can you say about the diagonal entries of J, J_{ii} for $i = 1, \ldots, n$? Explain.

   (b) Write down an example of a 2×2, non-zero, skew-symmetric matrix.

   (c) Does the set of 2×2 skew-symmetric matrices form a subspace of the vector space of all 2×2 matrices, $M_{2\times 2}$? Explain fully, using the definition of a subspace in your answer.

   (d) For your example in part (b) show that $v^T J v = 0$ for all $v \in \mathbb{R}^2$.

   (e) Show that $v^T J v = 0$ for all $v \in \mathbb{R}^n$, for any $n \times n$, skew-symmetric matrix J.

4. (30 points) For this problem, assume C is a 3×3 matrix with eigenvalues 0, 1 and 2. Five of the following six quantities can be computed from this information. Determine which five and find their values. In each case, briefly explain your reasoning.

   (a) the trace of C

   (b) the rank of C

   (c) the determinant of $C^T C$

   (d) the eigenvalues of $C^T C$ (Hint: What do the eigenvalues of $C^T C$ represent?)

   (e) the eigenvalues of $(C + I)^{-1}$ (Hint: What does the characteristic polynomial for $C + I$ tell you?)

   (f) the eigenvalues of C^3.

5. (30 points) Let $D = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$.

   (a) Give the definition for a matrix to be diagonalizable.

   (b) Find the diagonalization for matrix D.

   (c) Use part (b) to find D^5. (Use part (b), no credit if you just multiply D by itself 5 times!)

   (d) The most reasonable way to define the square root of a matrix, $A^{1/2}$, is to define $A^{1/2} = B$ when $A = B^2$. Use your answer in part (b) to find $D^{1/2}$.

6. (35 points) Let $A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$.

   (a) Compute the singular value decomposition of A.

   (b) Explain why the first r columns of U (or the columns of matrix P if you are using our text's formulation of SVD) give a basis for range(A).

   (c) Explain why the first r columns of V (or the columns of matrix Q) form a basis for corng(A).

   (d) Use the singular value decomposition of A to write A as the sum of two rank one matrices.

Extra Credit (up to 10 points). What is the most important thing (or useful to your major) you've learned this semester in this class?