

APPM 3310: Matrix Methods — Final Exam — December, 2008

On the front of your bluebook print (1) your name, (2) your student ID number, and (3) a grading table. **Explain all of your answers.** A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. **Start each problem on a new page.** No books, notes or electronic devices of any kind (e.g. cell phones, calculators, etc.) are permitted. **SHOW ALL WORK.**

1. (20 points) Let  $\mathcal{P}^{(3)}$  denote the *vector space* of all polynomials of degree less than or equal to 3.
  - (a) Are  $q_1(x) = x^2 + 1$ ,  $q_2(x) = x^2 - 1$  and  $q_3(x) = 7$  *linearly independent* elements of  $\mathcal{P}^{(3)}$ ?
  - (b) What is the *dimension* of  $V = \text{span}\{q_1, q_2, q_3\}$ ?
  - (c) Verify the *Triangle Inequality* for the functions  $q_1(x) = x^2 + 1$  and  $q_2(x) = x^2 - 1$  with respect to the  $L^2$ -norm on  $[0, 1]$ .
  
2. (20 points)
  - (a) Let  $V$  be an *inner product space*. State what it means for an  $n \times n$  matrix  $K$  to be the *Gramm matrix* associated to vectors  $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$ .
  - (b) Prove that every *positive definite*  $n \times n$  matrix  $K$  can be written as a Gramm matrix.
  - (c) Prove that a positive definite  $n \times n$  matrix  $K$  has positive *trace*:  $\text{tr}(K) > 0$ .
  
3. (30 points)
  - (a) Let  $P$  denote the *orthogonal projection* of  $\mathbb{R}^3$  onto the plane  $V = \{z = x + y\}$ . Find the matrix representation of  $P$ .
  - (b) Let  $Q$  denote the orthogonal projection of  $\mathbb{R}^3$  onto the plane  $W = \{z = x - y\}$ . Find the matrix representation of  $Q$ .
  - (c) Is the composition  $R = Q \circ P$  the same as the orthogonal projection of  $\mathbb{R}^3$  onto  $L = V \cap W$ ? Why or why not? Justify your answer.
  
4. (50 points)
  - (a) State the *Fundamental Theorem of Linear Algebra*. Be sure to include the hypothesis of the theorem in your answer.
  - (b) Let  $A$  be a real  $m \times n$  matrix. Show that the linear system  $Ax = b$  has a solution if and only if  $b$  is orthogonal to  $\text{coker}(A)$ . (*Hint*: If  $W$  is a finite-dimensional subspace of an inner product space, then  $(W^\perp)^\perp = W$ .)
  - (c) Find the dimensions of the four fundamental subspaces associated to  $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 4 & 1 \\ -1 & 0 & 1 \end{bmatrix}$ .
  - (d) Find a *basis* for  $\ker(A)$  and  $\text{rng}(A)$ .
  - (e) Find an *orthonormal basis* for  $\text{rng}(A)$ .
  - (f) Find a *QR* factorization of  $A$ .

**PLEASE TURN OVER. MORE PROBLEMS ON THE OTHER SIDE.**

5. (30 points) For this problem let  $A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ .

- (a) Find the *eigenvalues* and *eigenvectors* of the matrix  $A$ .
  - (b) Is  $A$  positive definite? Why or why not?
  - (c) Find an *orthonormal eigenvector basis* of  $\mathbb{R}^3$  determined by  $A$  or explain why none exists.
  - (d) Write out the *spectral factorization* of  $A$  if possible.
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6. (**Extra Credit** - 10 point bonanza)

Find the closest point on the plane spanned by  $(1, 1, 0, 0)^T$  and  $(0, 0, 1, 1)^T$  to  $\mathbf{b} = (3, 1, 2, 1)^T$ .  
What is the distance between  $\mathbf{b}$  and the plane?

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