APPM 3310: Matrix Methods — Exam #1 — February 20, 2008

On the front of your bluebook print (1) your name, (2) your student ID number, and (3) a grading table. **Explain all of your answers.** A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. No electronic devices of any kind (e.g. cell phones, calculators, etc.) are permitted.

Please sign your bluebook under the Honor Code to indicate that you have neither given nor received unauthorized assistance on this exam.

1. (40 points) For this problem, let $A = \begin{bmatrix} 1 & 3 & -5 & 2 \\ 2 & -1 & 1 & -4 \\ 4 & 5 & -9 & 0 \end{bmatrix}$

   (a) Find conditions on $\mathbf{b}$ so that $A\mathbf{x} = \mathbf{b}$ has (*i*) one, (*ii*) none, or (*iii*) infinitely many solutions.

   (b) Write your answer to part (a) in vector form, (i.e. $\mathbf{b} = (b_1, b_2, b_3)^T$) in order to find a basis for all of the $\mathbf{b}$ that have a solution to $A\mathbf{x} = \mathbf{b}$.

   (c) Find a basis for $\ker(A)$ and $\text{rng}(A)$.

   (d) Is the subspace formed by the span of the vectors in part (b) the same as $\text{rng}(A)$? Show this explicitly one way or the other.

   (e) Find the LU decomposition of A.

2. (40 points) This question is worth 40 points. You must do part (a). Then, answer 3 of the remaining 4 parts. (You can do all four for extra credit.)

   (a) Carefully define each of the four fundamental subspaces of an $m \times n$ matrix A. State the Fundamental Theorem of Linear Algebra, part 1.

   (b) Suppose A is an $m \times n$ matrix and B is a $p \times q$ matrix. If $\ker(A) = \ker(B)$ show that $\text{rank}(A) = \text{rank}(B)$. (Hint: How does part (a) help you?)

   (c) True or False (prove or find a counterexample): $\ker(A) \subset \ker(A^2)$ for a square $n \times n$ matrix A.

   (d) (*i*) Describe three different ways you could tell whether a matrix is singular or nonsingular. (*ii*) Now, suppose you know A is $n \times n$ and $A\mathbf{u} = A\mathbf{v}$ for some $\mathbf{u} \neq \mathbf{v}$. Is A singular or nonsingular? Explain.

   (e) Give the definition of symmetric matrix. If A and B are both symmetric, which of the following must be symmetric? (*i*) $A^2 - B^2$; (*ii*) $(A + B)(A - B)$; (*iii*) ABA.

3. (20 points) Let $V = \text{span}\{1, x^2, x^4\}$.

   (a) Is V a subspace of $\mathcal{P}^{(4)}$, where $\mathcal{P}^{(4)}$ is the vector space of all polynomials of degree less than or equal to 4?

   (b) Are the polynomials $p_1(x) = 1 + x^2$, $p_2(x) = 1 - x^2$, and $p_3(x) = 1 - x^4$ a basis for V?