Matrix Methods 3310 Sum 2006

Exam 1.

1. a)
$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -1 \\ -1 & 0 & -2 \end{bmatrix}$$

8 $$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -2 \end{bmatrix}$$

b) $LU\underline{x} = \underline{b}$ $\qquad L\underline{c} = \underline{b}$, $U\underline{x} = \underline{c}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \quad c_1 = 1,\ c_2 = 1,\ c_3 = -2$$

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$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \quad x_3 = 1,\ x_2 = -1,\ x_1 = 1$$

2. $$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & -2 \\ -1 & -2 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -2 & 0 & -1 \\ -1 & 0 & 0 & -2 \end{bmatrix}$$ 6

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ 10

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix} = PA$$ 4

3. $A\underline{x} = \underline{b}$, $PA\underline{x} = P\underline{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$

$LU\underline{x} = P\underline{b}$, $L\underline{c} = P\underline{b}$, $U\underline{x} = \underline{c}$ 4

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}, \quad c_1 = 4,\ c_2 = -3,\ c_3 = -2$$ 6

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 0 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix}$$

$x_4 = 1$, $-2x_2 - x_4 = -3$, $x_2 = 1$

$x_1 + x_2 + x_3 + x_4 = 4$, $x_1 = 4 - 1 - x_3 - 1 = 2 - x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 - x_3 \\ 1 \\ x_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \forall \alpha.$$ 10

4. $A = \begin{bmatrix} \underline{v}_1 & \underline{v}_2 & \underline{v}_3 & \underline{v}_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

span $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\} = \{A\underline{x} : \underline{x} \in \mathbb{R}^4\}$

Row reduction: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -2 & -1 & 0 \\ -2 & -2 & -1 & -2 \\ -1 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -2 & -1 & 0 \\ -2 & -1 & 0 & -2 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

pivot columns: 1, 2 & 4

Basis $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Range: From question 4: $\begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ 2

Corange: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}^T, \begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \end{bmatrix}^T, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}^T$ 4

Kernel: Solve for $A\underline{x} = \underline{0}$, i.e.

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 = 0, \quad 2x_2 + x_3 = 0$

$x_2 = -\frac{1}{2}x_3$

$x_1 = -x_2 - x_3 - x_4$

$= -\frac{1}{2}x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}x_3 \\ -\frac{1}{2}x_3 \\ x_3 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix}$$ 6

Cokernel: Different ways of finding it. Here I solve for $A\underline{x} = \underline{b}$ and find a condition on $\underline{b}$ so that $\underline{b} \in \text{range}(A)$. Since we know $\text{range}(A) \perp \text{cokernel}(A)$, this gives cokernel. 8

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 1 & -1 & 0 & 1 & b_2 \\ 2 & 0 & 1 & 0 & b_3 \\ 1 & 1 & 1 & 1 & b_4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 0 & -2 & -1 & 0 & -b_1+b_2 \\ 0 & -2 & -1 & -2 & -2b_1+b_3 \\ 0 & 0 & 0 & 0 & -b_1+b_4 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & b_1 \\ 0 & -2 & -1 & 0 & -b_1+b_2 \\ 0 & 0 & 0 & -2 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -b_1+b_4 \end{array}\right]$$

i.e. $\underline{b}$ satisfies $\begin{bmatrix} -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = 0$ Basis: $\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$