APPM3310 Matrix Methods

Semester Exam 2 **19 July 2006** **Total 100**

INSTRUCTIONS: You are allowed one, double sided A4 page with formulas. Calculators are not permitted. Work all problems and give the details of your calculations. A correct answer with the wrong argument may not receive any credit whereas a wrong answer with the correct argument might receive partial credit.

There are 5 questions, each question is worth 20 marks.

1. Test the following matrix for positive definiteness:

$$A = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}.$$

2. Calculate the Cholesky factorization of the following positive definite matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 8 & 0 \\ 1 & 0 & 3 \end{bmatrix}.$$

3. It is known that a physical process satisfies a law of the form $y = \alpha e^{-\beta x^2}$. Measurements of x and y produce the following table

x	0.1	0.2	0.3	0.4
y	1.3573	1.2281	1.1112	1.0055

Rewrite this in a form suitable for solving by linear least squares methods and find estimates for α and β.

4. Let

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}.$$

(a) Find the QR factorization of A. Hint: The columns of A are orthogonal.

(b) Make use of the QR factorization to solve the system $A\mathbf{x} = \mathbf{b}$ in a least squares sense where

$$\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) Calculate the reduced QR factorization of

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

using Householder matrices. Calculate R but you may write Q as the product of two Householder matrices without multiplying anything out, i.e. you may leave the Householder matrices in the form $H = I - 2\mathbf{u}\mathbf{u}^{\mathbf{T}}$, but give $\mathbf{u}$ explicitly. Explain why R is upper triangular in this case. Does this allows you to write down Q without needing to do the multiplications referred to above? If so, write down Q.

5. (a) If A is an $n \times n$ anti-symmetrix matrix, $A^T = -A$, show that $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} = \mathbf{0}$ for any $\mathbf{x}$.

   (b) A and B are $n \times n$ symmetric matrices. Show that $\mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x} = \mathbf{x}^{\mathbf{T}}\mathbf{B}\mathbf{x}$ for any $\mathbf{x}$ if and only if $A = B$. Hint: $A - B$ is also an symmetrix matrix.

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1. $A = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & -2 \end{bmatrix}$

First pivot is negative - not positive definite.

2. $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 8 & 0 \\ 1 & 0 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 \\ -2 & 4 & -2 \\ -1 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ -2 & 4 & -2 \\ -1 & 1/2 & 1 \end{bmatrix}, \text{ i.e. } A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

3.

x	0.1	0.2	0.3	0.4
x^2	0.01	0.04	0.09	0.16
y	1.3573	1.2281	1.1112	1.0055
$\ln y$	0.3055	0.2055	0.1054	0.0055

$y = \alpha e^{-\beta x^2}$

$\ln y = -\beta x^2 + \ln \alpha$

Solve $A\underline{x} = \underline{b}$ by least squares

where
$$A = \begin{bmatrix} 1 & -0.01 \\ 1 & -0.04 \\ 1 & -0.09 \\ 1 & -0.16 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0.3055 \\ 0.2055 \\ 0.1054 \\ 0.0055 \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} \ln\alpha \\ \beta \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 4 & -0.3 \\ 0.3 & 0.0354 \end{bmatrix}, \quad A^T \underline{b} = \begin{bmatrix} 0.6219 \\ -0.0216 \end{bmatrix}$$

$$\begin{bmatrix} \ln\alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0.3008 \\ 1.9381 \end{bmatrix} \qquad \beta = 1.9381, \quad \alpha = 1.351$$

4. a)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/2 & -1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

This is the reduced QR: Find 4th column $\perp$ first 3.

$$\begin{aligned} x + y + z + w &= 0 \\ x - y - z + w &= 0 \\ x \qquad\quad - w &= 0 \end{aligned}$$

$$x = w, \quad y + z = -2w \quad \Rightarrow \quad x = w = 0$$
$$-y - z = -2w \qquad\qquad y = -z$$

$$A = \begin{bmatrix} 1/2 & 1/2 & 1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 0 & 1/\sqrt{2} \\ 1/2 & -1/2 & 0 & -1/\sqrt{2} \\ 1/2 & 1/2 & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}$$

b) Use reduced form: $A^T A \underline{x} = A^T \underline{b}$

$$A = \tilde{Q}\tilde{R}$$

$$\tilde{R}^T \tilde{Q}^T \tilde{Q} \tilde{R} \underline{x} = \tilde{R}^T \tilde{Q}^T \underline{b}$$

$$\tilde{R}\,\underline{x} = \tilde{Q}^T \underline{b}$$

or $$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \\ 1/\sqrt{2} & 0 & 0 & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5/2 \\ 1/2 \\ 1/\sqrt{2} \end{bmatrix}$$

$z = 1/2, \quad y = 1/4, \quad x = 5/4$

c) $H = I - 2\underline{u}\,\underline{u}^T, \quad H\underline{x} = \underline{x} - 2\underline{u}\,\underline{u}^T\underline{x} = \alpha\,\underline{e}$

$\alpha = \|\underline{x}\|, \quad \underline{u} = (\underline{x} - \alpha\,\underline{e}) / \|\underline{x} - \alpha\,\underline{e}\|$

$$\underline{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \alpha = -\sqrt{3}, \quad \underline{u} = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \sqrt{3}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}{\| \cdot \|} = \begin{bmatrix} 1+\sqrt{3} \\ 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{6+2\sqrt{3}}}$$

$$HA = \begin{bmatrix} -\sqrt{3} & b_1 \\ 0 & b_2 \\ 0 & b_3 \end{bmatrix} \text{ with } \underline{b} = H \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{3+\sqrt{3}} \begin{bmatrix} 1+\sqrt{3} \\ 1 \\ 1 \end{bmatrix} 0 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Now $$\underline{u} = \frac{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + \sqrt{2}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}{\| \cdot \|} = \begin{bmatrix} 0 \\ 1+\sqrt{2} \\ -1 \end{bmatrix} \frac{1}{\sqrt{2+2\sqrt{2}}}$$

Columns orthogonal.

$$A = \begin{bmatrix} 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \end{bmatrix}.$$

5. a) If $A = -A^T$, then $\underline{x}^T A \underline{x} = (\underline{x}^T A \underline{x})^T$

$$= \underline{x}^T A^T \underline{x}$$

$$= -\underline{x}^T A \underline{x}$$

$$\Rightarrow \underline{x}^T A \underline{x} = 0$$

b) If $\underline{x}^T A \underline{x} = \underline{x}^T B \underline{x}$

then $\underline{x}^T (A - B) \underline{x} = 0$, for all $\underline{x}$.

Since $C = A - B$ is symmetric, write

$$q(\underline{x}) = \underline{x}^T C \underline{x}$$

$$= \sum_{ij} c_{ij} x_i x_j = 0 \text{ for all } \underline{x}$$

Choose $\underline{x}_{ij} = [0 \cdots \underset{i}{1}\, 0 \cdots \underset{j}{1} \cdots 0]$

i.e. 1 in i & jth position. Thus

$$q(\underline{x}_{ij}) = c_{ij} = 0$$