**APPM 3310: Matrix Methods — Exam #2 — November 12, 2008**

On the front of your bluebook print (1) your name, (2) your student ID number, and (3) a grading table. **Explain all of your answers.** A correct answer with no supporting work may receive no credit while an incorrect answer with some correct work may receive partial credit. **Start each problem on a new page.** No books, notes or electronic devices of any kind (e.g. cell phones, calculators, etc.) are permitted. **SHOW ALL WORK.**

1. (20 points)

   (a) Suppose $V$ is a complex vector space and that $\langle \mathbf{v}, \mathbf{w} \rangle$ is a pairing that takes two vectors $\mathbf{v}, \mathbf{w} \in V$ and produces a complex number $\langle \mathbf{v}, \mathbf{w} \rangle \in \mathbb{C}$. What three conditions must $\langle \mathbf{v}, \mathbf{w} \rangle$ satisfy to be considered a *complex inner product* on the vector space $V$? (Explicitly state your answer, don't just name the properties.)

   (b) Does $\langle \mathbf{v}, \mathbf{w} \rangle = v_1 \overline{w}_1 + 2 v_2 \overline{w}_2$ define a complex inner product on $\mathbb{C}^2$? (Show all work.)

   (c) Suppose $V$ is a vector space and suppose for each $\mathbf{v} \in V$ we have $\|\mathbf{v}\| \in \mathbb{R}$. What three conditions must $\|\mathbf{v}\|$ satisfy to be considered a *norm* on the vector space $V$?(Explicitly state your answer, don't just name the properties.)

2. (20 points)

   (a) State the *Cauchy-Schwarz Inequality.* When does equality hold?

   (b) State the *Triangle Inequality.* When does equality hold?

   (c) Verify the Cauchy-Schwarz inequality for the functions $f(x) = x$ and $g(x) = e^x$ with respect to the $L^2$ inner product on $[-1, 1]$.

3. (20 points)

   (a) Let $V$ be an inner product space. State what it means for an $n \times n$ matrix $K$ to be the *Gramm matrix* associated to $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$.

   (b) Under what conditions will the Gramm matrix $K$ be *positive definite*? Under what conditions will $K$ be *positive semi-definite*?

   (c) Prove that every positive definite $n \times n$ matrix $K$ can be written as a Gramm matrix.

4. (20 points)

   (a) Prove that a positive definite $n \times n$ matrix $K$ has positive determinant: $\det K > 0$.

   (b) Prove that every $2 \times 2$ symmetric matrix with positive determinant and postive trace is positive definite.

   (c) *True or False:* The set of complex vectors of the form $\begin{pmatrix} z \\ \overline{z} \end{pmatrix}$ for $z \in \mathbb{C}$ forms a *subspace* of $\mathbb{C}^2$. (Justify your answer.)

   (d) *True or False:* Reordering the original basis before starting the Gram-Schmidt process leads to the same orthogonal basis. (Justify your answer.)

5. (20 points) Find orthonormal bases for the four fundamental subspaces associated with the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ -1 & 2 & 0 & 1 \end{bmatrix}$.

6. (10 points) Find the closest vector, $\mathbf{v}^\star$, on the subspace spanned by $(0,0,1,1)^T$ and $(2,1,1,1)^T$ to the vector $\mathbf{b} = (0,3,1,2)^T$.