1) a) $A = \begin{pmatrix} 2 & 1 & 0 & 4 \\ 6 & 2 & 5 & 14 \\ -2 & -3 & 10 & 0 \end{pmatrix} \xrightarrow[③+①]{②-3①}$ (2 pts) (2 pts)

$\begin{pmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 5 & 2 \\ 0 & -2 & 10 & 4 \end{pmatrix} \xrightarrow{③-2①}$ (2pts

$U = \begin{pmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{U}$

b) $U = \begin{pmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

pivots

range$(A) = \left\{ \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} \right\}$

$\left(\begin{array}{cccc|c} 2 & 1 & 0 & 4 & 0 \\ 0 & -1 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$-x_2 + 5x_3 + 2x_4 = 0$
$x_2 = 5x_3 + 2x_4$

$2x_1 + x_2 + 4x_4 = 0$
$x_1 = \frac{1}{2}\left(-(5x_3 + 2x_4) - 4x_4\right)$
$x_1 = \frac{1}{2}(-5x_3 - 6x_4)$
$x_1 = -\frac{5}{2}x_3 - 3x_4$

$$\vec{x} = \begin{pmatrix} -\frac{5}{2}x_3 - 3x_4 \\ 5x_3 + 2x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -5/2 \\ 5 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix} x_4$$

$\ker(A) = \left\{ \begin{pmatrix} -5/2 \\ 5 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$

1c. rank A = # of pivots = 2

$\dim(\text{rng} A) = \dim(\text{corng} A) = \text{rank} A = 2$

$\dim(\text{Ker} A) = n - r = 2$

$\dim(\text{coker} A) = m - r = 1$

1d. Use $Lc = b$

$Ux = c$

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$$

$$\vec{L} \quad \vec{c} \quad = \vec{b}$$

$$\left.\begin{array}{l} a = 0 \\ b = 3 \\ c = 0 \end{array}\right\} \quad \vec{c} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 5 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

$$\vec{U} \quad \vec{x} \quad = \vec{c}$$

$2x_1 + x_2 + 4x_4 = 0$

$-1x_2 + 5x_3 + 2x_4 = 3$

$x_2 = -3 + 5x_3 + 2x_4$

$x_1 = \frac{3}{2} - \frac{5}{2}x_3 - 3x_4$

$$\vec{x} = \begin{pmatrix} 3/2 \\ -3 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5/2 \\ 5 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

2. a) $A, B \in M_{n\times n}$ then $C = A + B$ still has dimensions $n\times n$ so $C \in M_{n\times n}$. 5
And if $A \in M_{n\times n}$, $c \in \mathbb{R}$ then cA is still $n\times n$ so $cA \in M_{n\times n}$. 5

Can show w/ $a_{ij} + b_{ij} = c_{ij} \in \mathbb{R}$
and $ca_{ij} \in \mathbb{R}$
etc.

b) Basis

5 $\{E_{ij}\}_{i,j=1}^{n}$ where E_{ij} has a 1 in the ij^{th} spot and 0's everywhere else.

5 $\boxed{\text{Dim} = n^2}$

c) i) T $tr(A+B) = \sum_{i=1}^{n} a_{ii} + b_{ii} = \sum_{i=1}^{n} b_{ii} + a_{ii} = tr(B+A)$.

ii) NO. Let $A, B \in T$ so

$trA = 1$, $trB = 1$.

Then $tr(A+B) = trA + trB = 1 + 1 = 2$

So $A + B \notin T$.

OR let $c \in \mathbb{R}$ $c \neq 1$ then

$tr(cA) = c \cdot trA = c \cdot 1 = c \neq 1$.

3. a) TRUE

$A_{n \times n}$

$\to$ if rank $= r = n$

dim ker $A = r - n = r - r = 0$

therefore ker $A = \{\vec{0}\}$

$\to$ if ker $A = \{\vec{0}\}$

dim ker $A = 0 = r - n$

therefore $r = n =$ rank A

b) FALSE

zero vector belongs to all spans (independent and dependent)

c) False. If $n > m$ then could have free vars, but still span $\mathbb{R}^n$.

<u>Ex:</u> $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

spans $\mathbb{R}^2$

and has nonzero kernel.