1. QR decomposition of $A = \begin{bmatrix} 1 & -1 & 2 & 5 \\ 1 & 2 & 1 & -1 \\ 1 & -1 & 0 & -7 \\ 1 & 0 & 3 & 3 \end{bmatrix}$

Let $\vec{w}_i = i^{th}$ col of A

Then $\vec{w}_1 \cdot \vec{w}_2 = 0$, $\vec{w}_1 \cdot \vec{w}_4 = 0$ $\vec{w}_2 \cdot \vec{w}_4 = 0$ $\vec{w}_2 \cdot \vec{w}_3 = 0$

Only $w_1$ & $w_3$ And $w_3$ & $w_4$ are NOT $\perp$.

Use modified G-S

$$\vec{w}_1 = r_{11}\vec{u}_1$$
$$\vec{w}_2 = r_{12}\vec{u}_1 + r_{22}\vec{u}_2$$
$$\vec{w}_3 = r_{13}\vec{u}_1 + r_{23}\vec{u}_2 + r_{33}\vec{u}_3$$
$$\vec{w}_4 = r_{14}\vec{u}_1 + r_{24}\vec{u}_2 + r_{34}\vec{u}_3 + r_{44}\vec{u}_4$$

But orthogonality conditions above mean these eqns reduce to

$$\vec{w}_1 = r_{11}\vec{u}_1$$
$$\vec{w}_2 = r_{22}\vec{u}_2$$
$$\vec{w}_3 = r_{13}\vec{u}_1 + r_{33}\vec{u}_3$$
$$\vec{w}_4 = r_{34}\vec{u}_3 + r_{44}\vec{u}_4$$

$r_{11} = \|w_1\| = \sqrt{4} = 2$ $\qquad$ $r_{22} = \|\vec{w}_2\| = \sqrt{6}$

$\vec{u}_1 = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ $\qquad$ $\vec{u}_2 = \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\-1\\0\end{pmatrix}$

$\|\vec{w}_3\| = \sqrt{14}$ $\qquad$ $r_{13} = \langle \vec{w}_3, \vec{u}_1 \rangle = 1 + \frac{1}{2} + 0 + \frac{3}{2} = 3$

$r_{33} = \sqrt{14-9} = \sqrt{5}$

$$\vec{u}_3 = \frac{\vec{w}_3 - r_{13}u_1}{r_{33}} = \frac{\begin{pmatrix}2\\1\\0\\3\end{pmatrix} - \frac{3}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\begin{pmatrix}1/2\\-1/2\\-3/2\\3/2\end{pmatrix} = \frac{1}{2\sqrt{5}}\begin{pmatrix}1\\-1\\-3\\3\end{pmatrix}$$

$$\|\vec{w}_4\| = \sqrt{84}$$

$$r_{34} = \langle \vec{w}_4, \vec{u}_3 \rangle = \frac{5}{2\sqrt{5}} + \frac{1}{2\sqrt{5}} + \frac{21}{2\sqrt{5}} + \frac{9}{2\sqrt{5}} = \frac{36}{2\sqrt{5}} = \boxed{\frac{18}{\sqrt{5}}}$$

the only really gross part...

$$r_{44} = \sqrt{84 - \frac{18}{\sqrt{5}}}$$

can leave like this!

$$\vec{u}_4 = \frac{\vec{w}_4 - r_{34}\vec{u}_3}{r_{44}} = \frac{\begin{pmatrix} 5 \\ -1 \\ -7 \\ 3 \end{pmatrix} - \frac{18}{5}\begin{pmatrix} 1 \\ -1 \\ -3 \\ 3 \end{pmatrix}}{\sqrt{84 - \frac{18}{\sqrt{5}}}} = \frac{\begin{pmatrix} 5 \\ -1 \\ -7 \\ 3 \end{pmatrix} - \frac{9}{5}\begin{pmatrix} 1 \\ -1 \\ -3 \\ 3 \end{pmatrix}}{\sqrt{84 - \frac{18}{\sqrt{5}}}}$$

in the whole problem, but now it's done.

$$QR = \begin{pmatrix} 1/2 & -1/\sqrt{6} & \frac{1}{2\sqrt{5}} & \\ 1/2 & 2/\sqrt{6} & -1/2\sqrt{5} & \\ 1/2 & -1/\sqrt{6} & -3/2\sqrt{5} & \\ 1/2 & 0 & 3/2\sqrt{5} & \end{pmatrix} \begin{pmatrix} 2 & 0 & 3 & 0 \\ 0 & \sqrt{6} & 0 & 0 \\ 0 & 0 & \sqrt{5} & 18/\sqrt{5} \\ 0 & 0 & 0 & \sqrt{84 - \frac{18}{\sqrt{5}}} \end{pmatrix}$$

Not Bad at all.

2. $\mathbb{R}^3$ w/ dot product.

$$V = \text{span}\left\{ \underset{\vec{v}_1}{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}, \underset{\vec{v}_2}{\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}} \right\} \qquad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

a) Closest pt. in $V$ to $\vec{b}$.

Let $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 2 \end{pmatrix}$

Then $K = A^T A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$

$\vec{f} = A^T \vec{b} = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

Solve $K\vec{x} = \vec{f}$ $\qquad \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$

Gives $\quad 3x_1 = 3 \Rightarrow x_1^* = 1$
$\qquad\quad 6x_2 = 3 \Rightarrow x_2^* = \frac{1}{2}$ $\qquad$ so $\vec{x}^* = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$ coeff's for $\vec{v}_1, \vec{v}_2$

$$\vec{v}^* = x_1^* \vec{v}_1 + x_2^* \vec{v}_2 = 1 \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/2 \\ 2 \end{pmatrix}$$

$\boxed{\text{so } \vec{v}^* = \begin{pmatrix} 1/2 \\ -1/2 \\ 2 \end{pmatrix}}$ is closest pt. to $\vec{b}$ in $V$.

2 b) $d^* =$ dist. from $\vec{b}$ to $V$

$$d^* = \|\vec{b} - \vec{v}^*\| = \sqrt{\langle b - v^*, b - v^* \rangle}$$

$$= \sqrt{\left\langle \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} \right\rangle}$$

$$= \sqrt{\frac{1}{4} + \frac{1}{4}}$$

$$\boxed{d^* = \sqrt{\frac{1}{2}}}$$

$$b - v^* = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1/2 \\ -1/2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$v^* - b$ works too.

4

K symmetric

3. a) If $K$ is nonsingular show that $K > 0 \Rightarrow K^{-1} > 0$.

$K > 0 \Rightarrow$ by definition

$$\vec{x}^T K \vec{x} > 0 \quad \forall \vec{x} \in \mathbb{R}^n, \ \vec{x} \neq 0$$

$$\begin{aligned}
\vec{x}^T K \vec{x} &= \vec{x}^T K (K^{-1} K) \vec{x} \\
&= (\vec{x}^T K^T) K^{-1} (K\vec{x}) \\
&= (K\vec{x})^T K^{-1} (K\vec{x}) \\
&= \vec{y}^T K^{-1} \vec{y} \qquad \text{where } \vec{y} = K\vec{x}
\end{aligned}$$

so $\vec{y}^T K^{-1} \vec{y} > 0 \quad \forall \vec{y} \in \mathbb{R}, \ \vec{y} \neq 0$

■

b) False. a null direction means $\vec{x}^T K \vec{x} = 0$ for some $\vec{x} \neq \vec{0}$. i.e. $K$ positive semi-definite or $K \geq 0$.

Counterexample:

$K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ has a null direction $\vec{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$(1 \quad 1) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 - 1 = 0$$

But $K$ has L.I. cols so has $\ker = \{\vec{0}\}$.

3.c) monomials $1, x, x^2$ w/ $L^2[0,1]$. Find Gram matrix

$$\langle 1,1\rangle = \int_0^1 1\,dx = x\Big|_0^1 = 1$$

$$\langle 1,x\rangle = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\langle 1,x^2\rangle = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\langle x^2,x^2\rangle = \int_0^1 x^4 = \frac{x^5}{5}\Big|_0^1 = \frac{1}{5}$$

$$\langle x,x\rangle = \int_0^1 x^2 = \frac{1}{3}$$

$$\langle x,x^2\rangle = \int_0^1 x^3 = \frac{x^4}{4}\Big|_0^1 = \frac{1}{4}$$

$$K = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{pmatrix}.$$

d) If $K > 0 \Rightarrow$ it can be written as a Gram Matrix.

Let $\langle \vec{v}, \vec{w}\rangle = \vec{v}^T K \vec{w}$ be the inner prod. on $\mathbb{R}^n$. associated w/ $K$.

Then let $K_{ij}$ be $(i,j)$ entry in $K$, $e_i$ = $i^{th}$ standard basis vector on $\mathbb{R}^n$.

Then $K_{ij} = \vec{e}_i^{\,T} K \vec{e}_j = \langle \vec{e}_i, \vec{e}_j\rangle$

So $K = \left(\langle \vec{e}_i, \vec{e}_j\rangle\right)$ is Gram matrix

built from $\{\vec{e}_i\}_{i=1}^n$ and inner prod. given by $K$. ∎

3.e) $\vec{q}(\vec{x}) = 2x_1^2 + x_1x_2 - 2x_1x_3 + 2x_2^2 - 2x_2x_3 + 2x_3^2$

corresponds to

$$K = \begin{pmatrix} 2 & 1/2 & -1 \\ 1/2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

is pos. def (symmetric)

has row ech. form $\begin{pmatrix} 2 & 1/2 & -1 \\ 0 & 15/8 & -3/4 \\ 0 & 0 & 6/5 \end{pmatrix}$

So $D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 15/8 & 0 \\ 0 & 0 & 6/5 \end{pmatrix}$ $\quad L^T = \begin{pmatrix} 1 & \frac{1}{4} & -1/2 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{pmatrix}$

and $L = \begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1 & 0 \\ -1/2 & -2/5 & 1 \end{pmatrix}$

Then $\Delta$ of vars is $\vec{y} = L^T x = \begin{pmatrix} 1 & 1/4 & -1/2 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\vec{y} = \begin{pmatrix} x_1 + \frac{1}{4}x_2 - \frac{1}{2}x_3 \\ x_2 - 2/5\, x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

so change of vars gives

$$q(\vec{x}) = \vec{y}^T D \vec{y} = 2y_1^2 + \frac{15}{8}y_2^2 + \frac{6}{5}y_3^2$$

$$= 2\left(x_1 + \tfrac{1}{4}x_2 - \tfrac{1}{2}x_3\right)^2 + \frac{15}{8}\left(x_2 - \tfrac{2}{5}x_3\right)^2 + \tfrac{6}{5}x_3^2$$

4. a) No. Fails positivity. Counter ex...

Let $f(x)$ be any constant function.
Then $f'(x)=0$

So $\langle f, f\rangle = \int_{-1}^{1} 0^2\,dx = 0.$

but $f(x) \neq 0$.

b) Yes. $\|\vec{v}\|_1$ & $\|\vec{v}\|_2$ norms on $V$.

Then

$\|\vec{v}\| = \frac{1}{2}\left(\|\vec{v}\|_1 + \|\vec{v}\|_2\right)$ is a norm.

i) Positivity: $\|\vec{v}\|$ with $\vec{v} \neq 0$ positive b/c

$\|\vec{v}\|_1 > 0$ by $\| \; \|_1$ a norm
$\|\vec{v}_2\| > 0$ by $\| \; \|_2$ a norm
so $\frac{1}{2}(\|\vec{v}\|_1 + \|\vec{v}\|_2) > 0.$

ii) Homogeneity: $c \in \mathbb{R}$ then

$$\begin{aligned}
\|c\vec{v}\| &= \tfrac{1}{2}\left(\|c\vec{v}\|_1 + \|c\vec{v}\|_2\right) \\
&= \tfrac{1}{2}\left(|c|\,\|\vec{v}\|_1 + |c|\,\|\vec{v}\|_2\right) && \text{by } \| \; \|_1, \| \; \|_2 \text{ norms} \\
&= \frac{|c|}{2}\left(\|\vec{v}_1\| + \|\vec{v}\|_2\right) \\
&= |c| \cdot \|\vec{v}\| \ \blacksquare
\end{aligned}$$

ii) $\Delta$-Inequality

$$\begin{aligned}
\|\vec{v} + \vec{w}\| &= \tfrac{1}{2}\left(\|\vec{v}+\vec{w}\|_1 + \|\vec{v}+\vec{w}\|_2\right) \\
&\leq \tfrac{1}{2}\left(\|\vec{v}\|_1 + \|\vec{w}\|_1 + \|\vec{v}\|_2 + \|\vec{w}\|_2\right) && \leftarrow \text{by } \| \; \|_1 \,\&\, \| \; \|_2 \text{ norms} \\
&= \tfrac{1}{2}\left(\|\vec{v}\|_1 + \|\vec{v}\|_2\right) + \tfrac{1}{2}\left(\|\vec{w}\|_1 + \|\vec{w}\|_2\right) \\
&= \|\vec{v}\| + \|\vec{w}\| \ \blacksquare
\end{aligned}$$